A survey on Warped Cones

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Warped cones provide a bridge between dynamics and coarse geometry. This can be used to construct exotic metric spaces and families of expanders. Warped cones can also help bringing geometric intuition into dynamics matters. This poster is a brief survey, feel free to ask me for more details.

Definitions

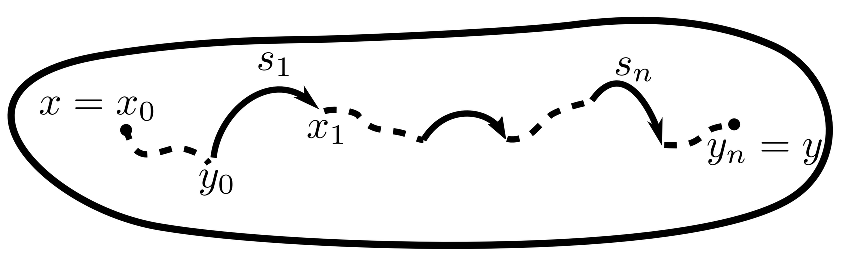
Let $\Gamma = \langle S \rangle$ be a finitely generated group with $S = s^{-1}$ and $1 \in S$. Let $\Gamma \curvearrowright (X, d)$ be an action by homeomorphisms on a compact metric space.

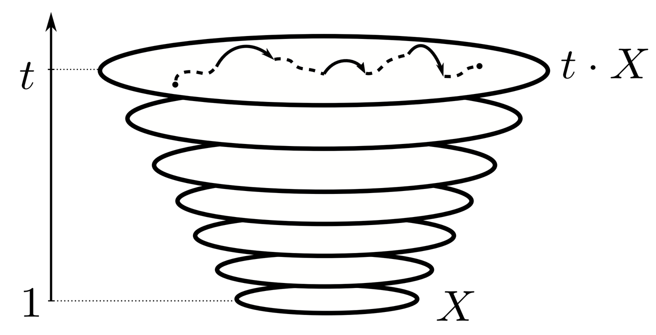
Definition. The *warped metric* on X is the largest metric δ_S such that $\delta_S(x, y) \leq d(x, y)$ for every $x, y \in X$ and $\delta_S(x, s(x)) \leq 1$ for every $s \in S$.

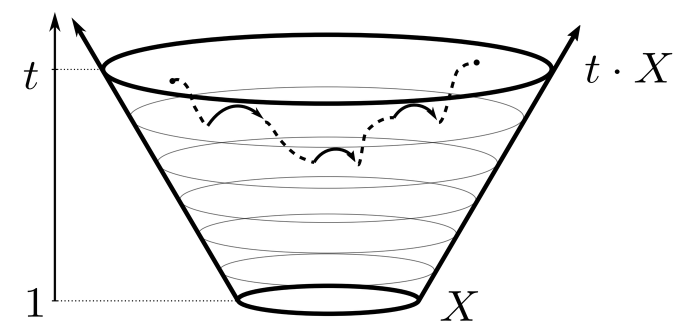
For every pair of points $x, y \in X$, their warped distance is equal to

$$\delta_S(x,y) = \inf_{n\in\mathbb{N}} \Big\{ n + \sum_{i=0}^{n} d(x_i, y_i) \;\Big|\; (x_0,y_0),\dots,(x_n,y_n) \text{ jumping-sequence from } x \text{ to } y \Big\}$$

where $(x_0, y_0), \dots, (x_n, y_n)$ is a *jumping-sequence* from x to y if $x = x_0$, $y = y_n$ and there exists $s_i \in S^{\pm}$ such that $x_i = s_i(y_{i-1})$ for every $0 < i \leq n$.







Definition. The *warped cone* $\mathcal{WC}(\Gamma \curvearrowright X)$ is the family of metric spaces $\{(X, \delta_S^t) \mid t \geq 1\}$, where δ_S^t is defined as the warping of the rescaled metric $t \cdot d$.

Equip $X \times [0, \infty)$ with a 'cone metric' *e.g.* defining $d_{\rm cone}((x,t),(x',t')) := \min\{t,t'\} d(x,x') + |t - t'|$.

Definition. The *unified warped cone* $\mathcal{UWC}(\Gamma \curvearrowright X)$ is $X \times [0, \infty)$ equipped with the warped metric.

Convention. We say that two families of metric spaces $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are coarsely equivalent if X_i and Y_i are uniformly quasi-isometric.

Using the above convention, the warped cone $\mathcal{WC}(\Gamma \curvearrowright X)$ is coarsely equivalent to the family of level sets $X \times \{t\}$ of the unified warped cone $\mathcal{UWC}(\Gamma \curvearrowright X)$.

Basic properties and examples

The definition of the warped metric depends on the choice of generating set, but different generating sets yield coarsely equivalent warped cones. Here are two basic lemmas:

Lemma. If two actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ are conjugate via a bi-Lipschitz homeomorphisms, then $\mathcal{WC}(\Gamma \curvearrowright X)$ and $\mathcal{WC}(\Gamma \curvearrowright Y)$ are bi-Lipschitz equivalent.

Lemma. If $F \trianglelefteq \Gamma$ is a finite normal subgroup, $\Gamma \curvearrowright X$ an action and $\Gamma/F \curvearrowright X/F$ the induced quotient action, then $\mathcal{WC}(\Gamma \curvearrowright X)$ and $\mathcal{WC}(\Gamma/F \curvearrowright Y/F)$ are coarsely equivalent.

Box spaces

Let $\Gamma_n \lhd \Gamma$ be a decreasing sequence of finite index normal subgroups, let $\mathcal{G}_n := {\rm Cay}(\Gamma/\Gamma_n, S)$ be the Cayley graphs of the quotients.

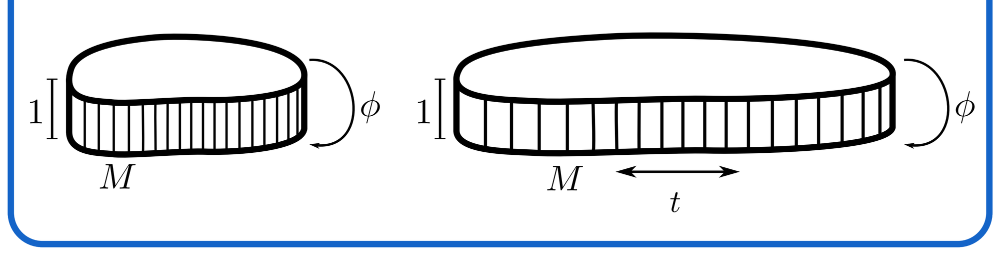
Definition. The *box space* $\Box_{(\Gamma_n)}\Gamma$ is defined imposing a metric on $\bigsqcup_n \mathcal{G}_n$ so that $d(\mathcal{G}_n, \mathcal{G}_m) \geq {\rm diam}(\mathcal{G}_n) + {\rm diam}(\mathcal{G}_m)$ for every n and m. The box space is well-defined up to coarse equivalence.

Let $\widehat{\Gamma}$ be the profinite completion w.r.t. Γ_n.

Lemma ([6]). It is possible to equip $\widehat{\Gamma}$ with a metric so that the box space $\Box_{(\Gamma_n)}\Gamma$ is coarsely equivalent to a sequence of level sets in $\mathcal{UWC}(\Gamma \curvearrowright \widehat{\Gamma})$.

Mapping tori

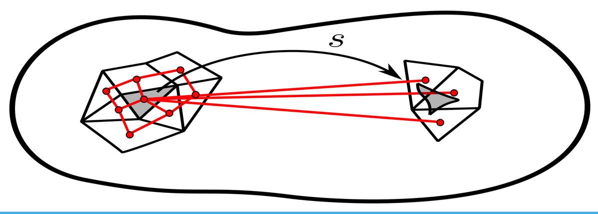
Let (M, d) be a closed Riemannian manifold and $\phi\colon M \to M$ a diffeomorphism. The product $M \times [0, 1]$ is foliated vertically. Let d^t be the metric obtained scaling by t the directions normal to the foliation. This induces a Riemannian metric on the mapping torus $M \times [0, 1]/(x, 1) \sim (\phi(x), 0)$ that makes it quasi-isometric to the level t of the warped cone $\mathcal{WC}(\phi\colon \mathbb{Z} \curvearrowright M)$.



Approximating graphs

Let (M, d) be a compact Riemannian manifold and $\Gamma \curvearrowright M$ an action by diffeomorphisms. For any $t \geq 1$, let $\mathcal{P}_t$ be a partition of M into connected regions of diameter $\approx 1/t$.

Definition. The *approximating graph* $\mathcal{G}_{\mathcal{P}_t}(\Gamma \curvearrowright M)$ is the graph having one vertex for each $R \in \mathcal{P}_t$ and so that $\{R, R'\}$ is an edge if and only if $s \cdot \overline{R} \cap \overline{R'} \neq \emptyset$ for some $s \in S$.



Lemma ([11]). The warped cone $\mathcal{WC}(\Gamma \curvearrowright M)$ is coarsely equivalent to the family of graphs $\mathcal{G}_{\mathcal{P}_t}(\Gamma \curvearrowright M)$.

A dictionary dynamics/coarse geometry

Here are sample results to illustrate the interplay between coarse geometry and dynamics:

Theorem ([5, 9]). A free bi-Lipschitz action $\Gamma \curvearrowright M$ on a compact Riemannian manifold is *topologically amenable* if and only if $\mathcal{UWC}(\Gamma \curvearrowright M)$ has *Property A*.

Corollary. Warped cones can produce exotic metric spaces with or without Property A.

Theorem ([11]). Any bi-Lipschitz action on a compact Riemannian manifold is *expanding in measure* if and only if the approximating graphs $\mathcal{G}_{\mathcal{P}_t}(\Gamma \curvearrowright M)$ have *Cheeger constants* uniformly bounded from below.

Corollary. Warped cones can construct families of *expander graphs*.

Theorem ([4, 7, 1]). For any Banach space E, a measure preserving bi-Lipschitz action on a compact Riemannian manifold has *E-valued spectral gap* if and only if the approximating graphs $\mathcal{G}_{\mathcal{P}_t}(\Gamma \curvearrowright M)$ satisfy a uniform *E-valued Poincaré inequality*.

Corollary. Warped cones can construct families of *superexpander graphs*.

Geometry of warped cones

Here are some results regarding the geometry of warped cones:

Theorem ([1]). If $\Gamma \curvearrowright M$ is an essentially free isometric action on a compact Riemannian manifold, then the 'local coarse geometry' of the level sets in $\mathcal{WC}(\Gamma \curvearrowright M)$ is modelled on $\mathbb{R}^{\dim(M)} \times {\rm Cay}(\Gamma, S)$.

Corollary. Can produce countably many coarsely distinct superexpanders.

Theorem ([2]). Let $\Gamma < G$ and $\Lambda < H$ be torsion free, dense subgroups of compact Lie groups. If $\mathcal{WC}(\Gamma \curvearrowright G)$ and $\mathcal{WC}(\Lambda \curvearrowright H)$ are coarsely equivalent, then (assuming few extra hypotheses) the actions are conjugate by an (affine) diffeomorphism.

Corollary. There exists a continuum of coarsely distinct superexpanders.

Theorem ([12]). For any continuous action $F_S \curvearrowright M$ on a compact manifold, the *coarse fundamental group* of $\mathcal{WC}(F_S \curvearrowright M)$ is an (explicit) quotient of $\pi_1(M) \rtimes F_S$.

Corollary. There exist coarsely simply connected expanders. Such expanders are not coarsely equivalent to any box space.

Work in progress

This section is concerned with some work in progress with Kang Li and Jiawen Zhang. It focuses on weakenings of the notions of expander graphs and expansion in measure.

Theorem. A continuous action $\Gamma \curvearrowright M$ on a Riemannian manifold is *weakly expanding in measure* if and only if the approximating graphs $\mathcal{G}_{\mathcal{P}_t}(\Gamma \curvearrowright M)$ are *asymptotic expanders*.

The above inspired the following:

Theorem. A measure-class preserving action on a measure space X is strongly ergodic if and only if X admits an exhaustion by *domains of expansion*.

This generalizes a result of Marrakchi [3].

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