Expanders, Approximating Graphs and Warped Cones

Federico Vigolo

Supervisor: Cornelia Druţu

University of Oxford

Expanders

The *Cheeger constant* of a finite graph G is $h(G) \coloneqq \min |\partial A|/|A|$ where A is a set of vertices with $|A| \leqslant |G|/2$ and ∂A is the set edges exiting from A.

Definition. A *family of expanders* is a sequence of finite graphs G_n with uniformly bounded degree so that $|G_n| \to \infty$ and $h(G_n) \geqslant \varepsilon$ for a fixed constant $\varepsilon > 0$.

It is a theorem of Pinsker that random sequences of finite graphs are expanders, but it is generally hard to produce concrete examples. The first explicit families were found by Margulis:

Theorem (Margulis). *If $\Gamma = \langle S \rangle$ is a fin. gen. group with Kazhdan Property (T) and $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \cdots$ is a sequence of finite index subgroups, then they Cayley graphs $\mathrm{Cay}(\Gamma/\Gamma_n, S)$ form a sequence of expanders.*

A far reaching property of expander graphs is the following:

Theorem (Matoušec). *If the graphs G_n form a sequence of expanders then they do not coarsely embed in a Hilbert space uniformly.*

Approximating Graphs

Let $\Gamma = \langle S \rangle$ be a fin. gen. group acting on a space X, and let $\mathcal{P}$ be a finite partition of X.

Definition. The graph approximating the action $\Gamma \curvearrowright X$ with the partition $\mathcal{P}$ is the graph $\mathcal{G}(\mathcal{P})$ with:

- vertices $\longleftrightarrow$ regions $R \in \mathcal{P}$;
- (R, R') is an edge $\Leftrightarrow$ $s(R) \cap R' \neq \emptyset$ for some $s \in S$.

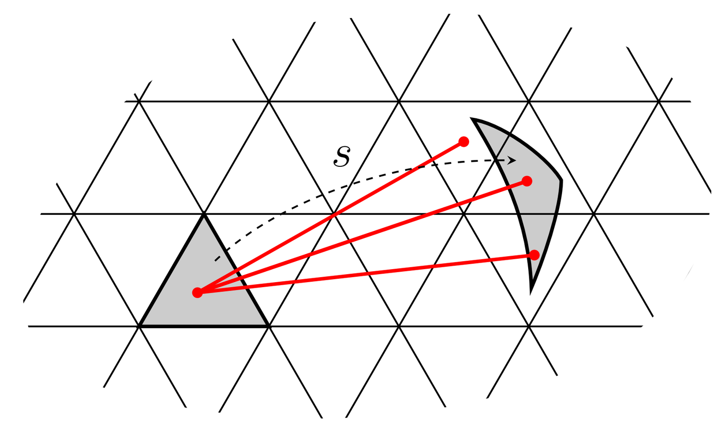

Fig. 1: Edges of the approximating graph.

Warped Cones

Let $\Gamma = \langle S \rangle$ be a fin. gen. group acting on a metric space (X, d). Let $\big(X \times [1, \infty), d_{\mathcal{C}}\big)$ be the Euclidean cone over (X, d), *i.e.* $d_{\mathcal{C}}$ is the distance so that the path metric induced on the level set $X \times \{t\}$ is equal to $t \cdot d$.

Definition. The *warped cone* is the space $\mathcal{O}_\Gamma X \coloneqq \big(X \times [1, \infty), \delta_\Gamma\big)$ where δ_Γ is the largest distance so that $\delta_\Gamma \leqslant d_{\mathcal{C}}$ and $\delta_\Gamma(xt, s(x)t) \leqslant 1$ for every $x \in X$, $t \in [1, \infty)$ and $s \in S$.

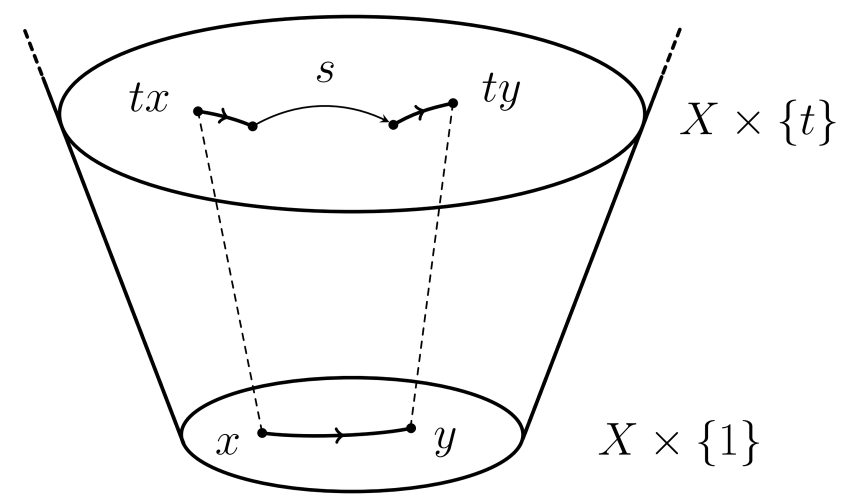

Fig. 2: Geodesic paths in a warped cone.

Some results

Let $\Gamma = \langle S \rangle$ with $1 \in S = S^{-1}$ finite and let (X, ν) be a probability space. We say that an action $\Gamma \curvearrowright (X, \nu)$ is *expanding in measure* if there exists an $\alpha > 0$ so that $\nu\big(S \cdot A\big) \geqslant (1+\alpha)\nu(A)$ for every $A \subset X$ with $\nu(A) \leqslant 1/2$.

Theorem. Let $\Gamma \curvearrowright (X, d, \nu)$ be a nice[1] action and let $\mathcal{P}_n$ be a sequence of nice[2] partitions of X with mesh tending to 0. Then the approximating graphs $\mathcal{G}(\mathcal{P}_n)$ form a family of expanders if and only if the action is expanding in measure.

Corollary. Assume $a, b \in \mathrm{SO}(3, \overline{\mathbb{Q}})$ generate a free group. Then the graphs approximating their action by rotations on $\mathbb{S}^2$ are expanders.

[1] ν is a Radon measure, (X, d, ν) is doubling and the action is by quasi-symmetric maps with bounded measure distortion.

[2] The regions of $\mathcal{P}_n$ have uniformly bounded measure ratios and eccentricity.

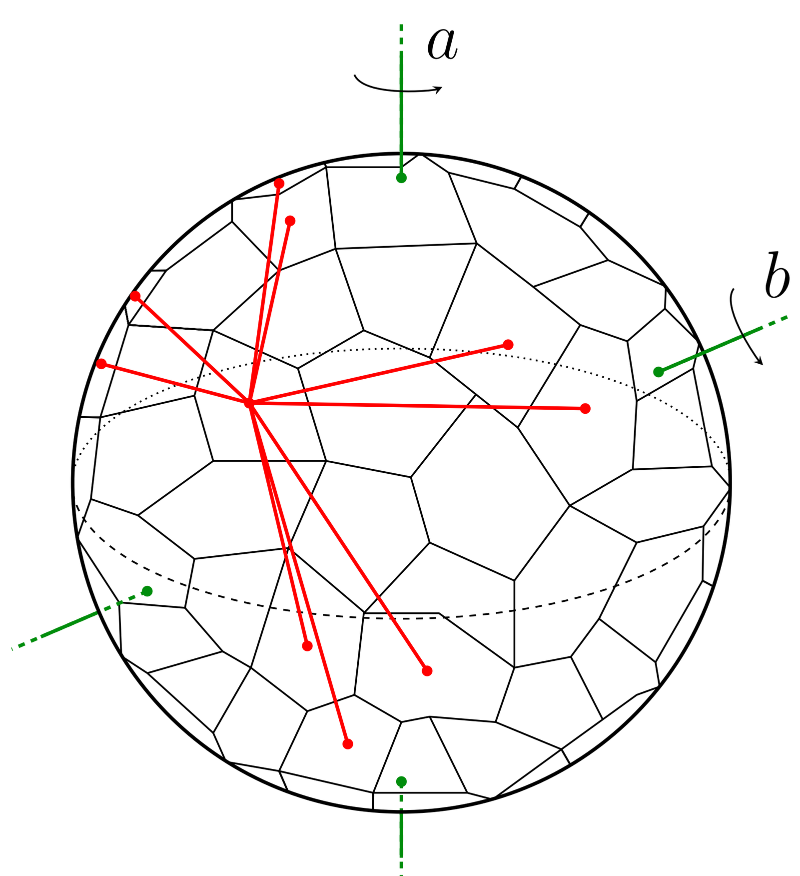

Fig. 3: Expanders out of actions by rotations.

Proposition. If the mesh of $\mathcal{P}_n$ is $1/n$, then the graphs $\mathcal{G}(\mathcal{P}_n)$ are uniformly quasi isometric to the level sets $X \times \{n\} \subset \mathcal{O}_\Gamma X$.

Corollary.[3] If the action $\Gamma \curvearrowright X$ is expanding in measure, then the warped cone $\mathcal{O}_\Gamma X$ does not coarsely embed into an Hilbert space.

The expanders arising as approximating graphs are genuinely different from the examples built using Cayley graphs. Indeed, we have the following:

Theorem. Sequences of expanders of the form $\mathcal{G}(\mathcal{P}_n)$ and $\mathrm{Cay}(\Gamma/\Gamma_n, S)$ cannot be uniformly coarsely equivalent.

[3] This result was also proved by Nowak and Sawicki by different means.