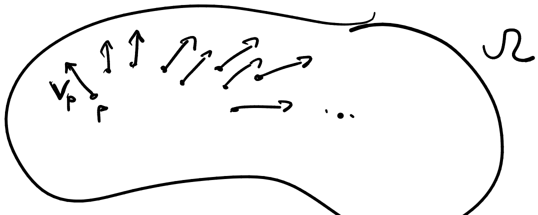
— Crash course on planar differential geometry —

<u>References:</u> [Gallot-Hulin-Lafontaine: Riemannian geometry]
[do Carmo: differential geometry of curves and surfaces]

N.B. this is not meant to be a precise course, I am only trying to give an idea of the differential geometric background needed to formalize the theory of surfaces. I will be sweeping some details under the carpet.

$\Omega \subset \mathbb{R}^2$ domain. $\mathbb{E}^2 := \mathbb{R}^2$ with the <u>Euclidean metric</u>

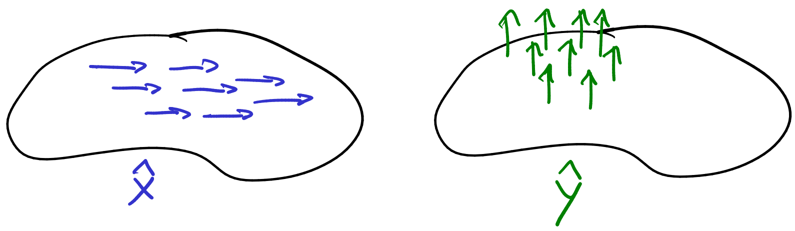
Working definition of "<u>vector field</u>": a smooth function that associates to each $p \in \Omega$ a vector $V_p \in \mathbb{R}^2$. We say that V_p is a vector at p.



$T_p\Omega := \{\text{vectors based at } p\}$

$T\Omega := \{\text{vector fields}\}$

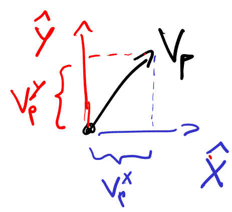
We have two <u>coordinate vector fields</u>: $\hat{X} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\hat{Y} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Every vector field can be naturally expressed in terms of these coordinate vector fields. We will use the following notation interchangeably:

$V \in T\Omega$ is a section of the tangent bundle, i.e. a function

$$V : \Omega \longrightarrow T\Omega$$
$$p \longmapsto V_p =: V(p) \quad (= V(x,y) \text{ if } p = (x,y))$$



$$V_p = \begin{pmatrix} V_p^x \\ V_p^y \end{pmatrix} = V_p^x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + V_p^y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = V_p^x \hat{X}_p + V_p^y \hat{Y}_p$$

I.e. $V = V^x \hat{X} + V^y \hat{Y}$ where $V^x, V^y : \Omega \longrightarrow \mathbb{R}$ are smooth functions

$$V^x(x,y) = V^x(p) = V^x_p$$
$$p = (x,y)$$

Vector fields can be identified with derivations. That is, for every differentiable function f defined in a neighbourhood of $p \in \Omega$ and every $V_p \in T_p\Omega$ it makes sense to consider the partial derivative of f in direction V_p:

$$\frac{\partial f}{\partial V_p} = V_p^x \frac{\partial f}{\partial x} + V_p^y \frac{\partial f}{\partial y} \in \mathbb{R}$$

these are the usual partial derivatives (and, by definition, they coincide with $\frac{\partial f}{\partial \hat{X}_p}, \frac{\partial f}{\partial \hat{Y}_p}$ respectively)

As f varyies, the values $\frac{\partial f}{\partial V_p}$ uniquely determine V_p.
For this reason vector fields are indeed identified with derivations

More precisely, a derivation is a $\mathbb{R}$-linear map $\delta : C^\infty(\Omega) \to C^\infty(\Omega)$ that satisfies the Leibnitz rule: $\delta(fg) = f\,\delta(g) + \delta(f)\,g$

Fact/Exercise: There is a 1-1 correspondence $T\Omega \longleftrightarrow \{\text{derivations}\}$

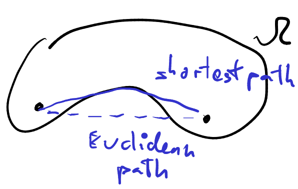
All of this is just plain old differential geometry. To start doing some actual geometry (ie Riemannian geometry) we need to choose inner products $\langle\cdot,\cdot\rangle_p$ varying smoothly with p. Since $\hat{X}_p$ & $\hat{Y}_p$ are a basis, we can use it to identify the scalar product $\langle\cdot,\cdot\rangle_p$ with a matrix $\begin{pmatrix} E_p & F_p \\ F_p & G_p \end{pmatrix}$, so that

$$\langle \hat{X}_p, \hat{X}_p \rangle = E_p \qquad \langle \hat{X}_p, \hat{Y}_p \rangle_p = \langle \hat{Y}_p, \hat{X}_p \rangle_p = F_p \qquad \langle \hat{Y}_p, \hat{Y}_p \rangle_p = G_p$$

In general, $\langle V, W \rangle_p := \langle V_p, W_p \rangle_p = E_p V_p^x W_p^x + F_p (V_p^x W_p^y + V_p^y W_p^x) + G_p V_p^y W_p^y$

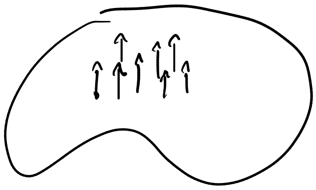
The Euclidean metric is obtained letting $E \equiv 1 \equiv G$ and $F \equiv 0$

Rmk the Euclidean metric on $\Omega \subset \mathbb{R}^2$ coincides with the restriction of the metric on $\mathbb{E}^2$ if Ω is convex



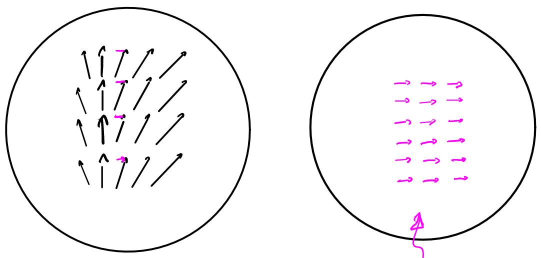
How about derivating other stuff, e.g. other vector fields?

If Ω is equipped with the Euclidean metric it's quite obvious what we need to do:



V constant vector field $\rightsquigarrow$ "$\dfrac{\partial V}{\partial W} \equiv 0$" $\forall W$.

(the quotation mark is because you'd better not getting used to this notation)



difference as x grows

$$V(x,y) = x\,\hat{X}(x,y) + \hat{Y}(x,y)$$

$$\rightsquigarrow \text{"}\frac{\partial V}{\partial \hat{X}} \equiv \hat{X}\text{"} \qquad \text{"}\frac{\partial V}{\partial \hat{Y}} \equiv 0\text{"}$$

$$\text{"}\frac{\partial V}{\partial W} = W^x \frac{\partial V}{\partial \hat{X}} + W^y \frac{\partial V}{\partial \hat{Y}}\text{"} \quad \text{etc.}$$

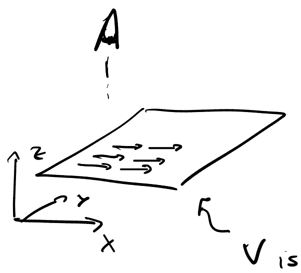
Long story short: the natural definition (in coordinates) is

$$\text{"}\frac{\partial V}{\partial \hat{X}}\text{"} = \frac{\partial V^x}{\partial x}\hat{X} + \frac{\partial V^y}{\partial x}\hat{Y} \qquad \text{"}\frac{\partial V}{\partial \hat{Y}}\text{"} = \frac{\partial V^x}{\partial y}\hat{X} + \frac{\partial V^y}{\partial y}\hat{Y}$$

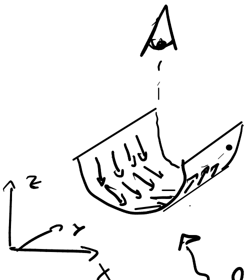
$$\text{"}\frac{\partial V}{\partial W}\text{"} = \left(W^x \frac{\partial V^x}{\partial x} + W^y \frac{\partial V^x}{\partial y}\right)\hat{X} + \left(W^x \frac{\partial V^y}{\partial x} + W^y \frac{\partial V^y}{\partial y}\right)\hat{Y}$$

Bad News: this formula is only adequate when Ω is equipped with the Euclidean metric. A general notion of "derivation of vector fields" ought to hold the Riemannian metrics into consideration.

Eg:

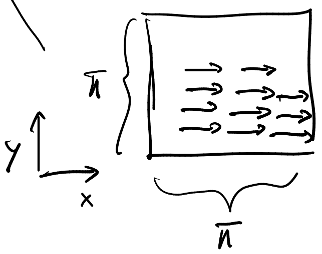


V is the constant vector field

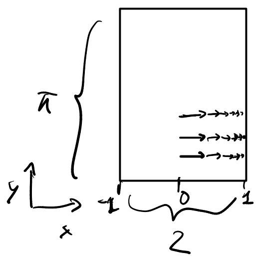


from the surface's perspective nothing has changed: V is still the constant vector field

look from above



$V = \hat{X}$



$$E(x,y) = \frac{1}{\sin^2(x)}$$

$$F \equiv 0 \qquad G \equiv 1$$

$$V(x,y) = \sin(x)\,\hat{X}$$

Now if we just take derivatives we get something non-trivial, which is bad

Ok, so how do we define a good notion of derivative of vector fields?

Given $V, W \in T\Omega$, we want to define $D_V W$ (derivative of W in direction of V)

We want
- $D_V W \in \Omega$
- $D_V(a_1 W_1 + a_2 W_2) = a_1 D_V W_1 + a_2 D_V W_2$ $\quad \forall a_1, a_2 \in \mathbb{R}$ (Linear).
- $D_V fW = \frac{\partial f}{\partial V} W + f D_V W$ $\quad \forall f: \Omega \to \mathbb{R}$ (Leibnitz rule)
- $D_{a_1 V_1 + a_2 V_2} W = a_1 D_{V_1} W + a_2 D_{V_2} W$ $\quad$ (bi)-linear
- $D_{fV} W = f D_V W$

This is not Leibnitz.

This condition is saying that the derivative is defined pointwise (as opposed to locally) in V.

That is, given $V_p \in T_p W$ it makes sense to ask how W is changing in direction V_p, even without knowing how V is defined in a neighbourhood of p.

An operator $D: T\Omega \to T\Omega$ satisfying these conditions is called connection

There are many connections, we need to add extra restrictions

We are used from multivariable calculus that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, we thus require that our connection D satisfies

$$D_{\hat{x}} \hat{y} = D_{\hat{y}} \hat{x}.$$

Such a connection is called torsion free

Warning! This does not imply $D_V W = D_W V$ $\quad \forall V, W \in T\Omega$.
this definition relies on the fact that we fixed a good system of coordinates x, y

[As a matter of fact, Torsion freeness is not defined this way in general.
To obtain a coordinate-free definition, we start by noting that $[V, W]$ is a well-defined vector field. That is, we can compose derivations $\frac{\partial}{\partial V} \circ \frac{\partial}{\partial W} : C^\infty(\Omega) \to C^\infty(\Omega)$ yet, $\frac{\partial}{\partial V} \circ \frac{\partial}{\partial W}$ is not a derivation

On the contrary, the difference $\frac{\partial}{\partial V}\circ\frac{\partial}{\partial W} - \frac{\partial}{\partial W}\circ\frac{\partial}{\partial V}$ is a derivation (easy exercise)

It follows from the Fact/Exercise that $\frac{\partial}{\partial V}\circ\frac{\partial}{\partial W} - \frac{\partial}{\partial W}\circ\frac{\partial}{\partial V}$ identifies a vector field.
We define $[V,W] \in T\Omega$ to be that vector field.

Def the connection D is torsion free if $D_V W - D_W V = [V,W]$ $\forall V,W \in T\Omega$

(not fun) exercise Check that, with our coordinate system, $D_{\hat{X}}\hat{Y} = D_{\hat{Y}}\hat{X}$
implies $D_V W - D_W V = [V,W]$

We still have not use the metric anywhere: the definition of torsion-free connection is purely differential. It is now time to make use of $\langle\cdot,\cdot\rangle_p$

Def A connection D is compatible with $\langle\cdot,\cdot\rangle_p$ iff

$$(*) \qquad \frac{\partial}{\partial U}\langle V,W\rangle = \langle D_U V, W\rangle + \langle V, D_U W\rangle \qquad \forall U,V,W \in T\Omega$$

(Note that our naive approach to derivation of vector fields was indeed compatible with the Euclidean scalar product)

Thm There exists a unique torsion free connection compatible with $\langle\cdot,\cdot\rangle$
This is called the Levi-Civita connection

Sketch of pf cycling U,V,W in $(*)$ and adding or subtracting, we get

$$2\langle D_U V, W\rangle = \boxed{\frac{\partial}{\partial U}\langle V,W\rangle + \frac{\partial}{\partial V}\langle W,U\rangle - \frac{\partial}{\partial W}\langle U,V\rangle + \langle [U,V],W\rangle - \langle [V,W],U\rangle - \langle [W,U],V\rangle} \qquad ⭐$$

all of this is uniquely defined

and one also checks that this expression is enough to define a vector field $D_U V$
(exercise) ∎

If one wants to do computations, it may be convenient to write everything in coordinates. In particular

$$D_{\hat{x}}\hat{x} = \Gamma_{xx}^{x}\hat{x} + \Gamma_{xx}^{y}\hat{y}$$
$$D_{\hat{x}}\hat{y} = \Gamma_{xy}^{x}\hat{x} + \Gamma_{xy}^{y}\hat{y}$$
$$D_{\hat{y}}\hat{x} = \Gamma_{yx}^{x}\hat{x} + \Gamma_{yx}^{y}\hat{y}$$
$$D_{\hat{y}}\hat{y} = \Gamma_{yy}^{x}\hat{x} + \Gamma_{yy}^{y}\hat{y}$$

for appropriate coefficients $\Gamma_{jk}^{i}: M \longrightarrow \mathbb{R}$. These coefficients are called **Christoffel symbols**

Using the expression $\star$, one can compute them explicitely. [GHL, Proposition 2,54]
In general the expression is horrendous. Still, since we're using coordinats such that $[\hat{x},\hat{x}]=[\hat{x},\hat{y}]=[\hat{y},\hat{y}]=0$ things become marginally better, but still ugly.

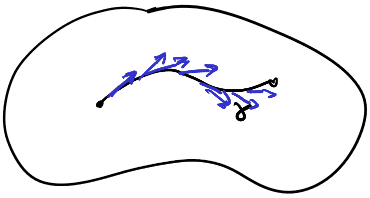
On the other hand we are mainly interested into metrics that are conformally equivalent to the Euclidean one, i.e. $\langle\cdot,\cdot\rangle = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ s.t. $F=0$ and $E=G=\lambda(x,y)$

this makes the computation worth doing because now $\hat{x}$ and $\hat{y}$ are orthogonal and we have
$$\langle V,\hat{x}\rangle = \lambda V^{x} \qquad \forall V \in T\Omega$$
$$\langle V,\hat{y}\rangle = \lambda V^{y}$$

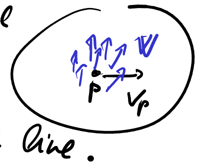
computations $\rightsquigarrow$

$$\boxed{\begin{array}{ll} \Gamma_{xx}^{x} = \Gamma_{xy}^{y} = \Gamma_{yx}^{y} = \frac{1}{2\lambda}\frac{\partial\lambda}{\partial x} & \Gamma_{yy}^{x} = -\frac{1}{2\lambda}\frac{\partial\lambda}{\partial x} \\ \Gamma_{yy}^{y} = \Gamma_{yx}^{x} = \Gamma_{xy}^{x} = \frac{1}{2\lambda}\frac{\partial\lambda}{\partial y} & \Gamma_{xx}^{y} = -\frac{1}{2\lambda}\frac{\partial\lambda}{\partial y} \end{array}}$$

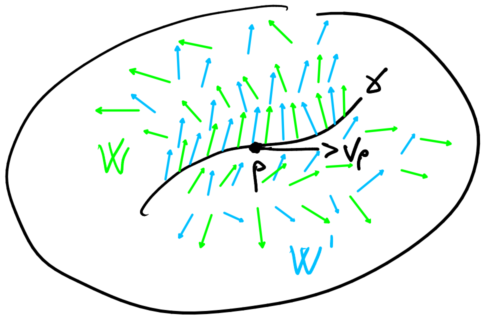
Ok, now we know how to take "derivatives" of vector fields.
Why did we do it? Because we want to know what it means to "go straight" in (Ω, d). That is, we want to know if the "velocity vector" of a curve $\gamma : [0,1] \longrightarrow \Omega$ is changing or not.



Small issue: the derivation $D_{V_p} W$ is defined only if W is define on a neighbourhood of p.
but we want to differentiate the "velocity vector" $\dot{\gamma}(t)$ which is only defined on a line.



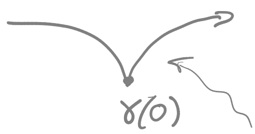
Solution: One can check (exercise) that if $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \Omega$ is smooth and so that $\gamma(0) = p$ & $\dot{\gamma}(0) = V_p$
and W, W' are two vector fields so that $W_{\gamma(t)} = W'_{\gamma(t)}$ $\forall t$,
then $D_{V_p} W = D_{V_p} W'$. That is, the "directional derivative" only measures how the vector field changes in <u>that</u> direction



In particular, if W is a vector field that is defined only on γ and it admits an extension $\widetilde{W}$ that is defined on a <u>neighbourhood</u> of $\gamma(0)$, then we define
$D_{\dot{\gamma}(0)} W := D_{\dot{\gamma}(0)} \widetilde{W}$ and this is well-defined.

<u>Rmk</u> if $\dot{\gamma}(0) \neq 0$ and the map $t \longrightarrow W_{\gamma(t)}$ is smooth, then it is always possible to find such an extension $\widetilde{W}$

(This is not always true if $\dot{\gamma}(0) = 0$:
One here needs to be more careful [see GHL Th 2.68]



can cause issues.)

this can be a smooth curve if $\dot{\gamma}(0) = 0$

More precisely, one does as follows: $\gamma: [0,1] \longrightarrow \Omega$ smooth curve, a vector field along γ is a map $W: [0,1] \longrightarrow T\Omega$ st $W(t) \in T_{\gamma(t)}\Omega \quad \forall t.$

Writing $W(t) = W^x(t)\,\hat{X} + W^y(t)\,\hat{Y}$, and $\gamma(t) = (\gamma^x(t), \gamma^y(t))$ we can compute $D_{\dot{\gamma}(t)} W(t)$

using the Christoffel symbols and linearity + Leibnitz

this is another vector field along γ

If you do the computation, it turns out that

$$\boxed{D_{\dot{\gamma}(t)} W(t) = \sum_i \left[\frac{dW^i(t)}{dt} + \sum_{jk} \Gamma^i_{jk} \frac{d\gamma^j(t)}{dt} W^k(t) \right] \hat{I} \quad (\heartsuit)}$$

where $i,j,k = x$ or y and, consequently $\hat{I} = \hat{X}$ or $\hat{Y}$

Bottom line: if $\gamma: [0,1] \longrightarrow \Omega$ is a smooth curve st. $\dot{\gamma}(t) \neq 0 \ \forall t$ and W is a vector field defined on (the image of) γ, then the derivation $D_{\dot{\gamma}(t)} W$ is well-defined

In particular, $\dot{\gamma}(t)$ itself is a vector field defined on γ and we can thus consider the derivation $D_{\dot{\gamma}(t)} \dot{\gamma}$

Def A Riemannian geodesic is a smooth curve $\gamma: [0,1] \longrightarrow (\Omega, d)$ that has constant non-zero speed $\|\dot{\gamma}(t)\|_{\gamma(t)} \equiv C > 0$ and so that $D_{\dot{\gamma}(t)} \dot{\gamma}(t) \equiv 0$

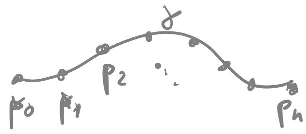
(i.e a constant speed curve that goes straight)

Since the theory makes sense, the following is true:

Thm a continuous curve $\gamma : [0,1] \longrightarrow (\Omega, d)$ is locally minimizing if and only if it is smooth and (its constant-speed reparametrization) is a geodesic.

(i.e. $\forall\, t \in [0,1]$ there is an $\varepsilon > 0$ such that
$d(\gamma(t-\varepsilon), \gamma(t+\varepsilon)) =$ length of $\gamma|_{[t-\varepsilon, t+\varepsilon]}$)

[the length of a continuous curve is defined as $\sup_{n, p_i} \left(\sum_{i=0}^{n-1} d(p_i, p_{i+1}) \right)$ where p_i are consecutive points on γ.



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Well, remember that we wrote explicitely (♡) what $D_v W$ is. We can specialize this to $D_{\dot\gamma(0)} \dot\gamma(0)$. Further, we also assume that $\langle \cdot, \cdot \rangle_p = \lambda(p) \cdot \text{Euclidean}$

So, let $\gamma(t) = (\gamma^x(t), \gamma^y(t))$, so that $\dot\gamma^x(t) = \frac{d\gamma^x(t)}{dt}$ $\dot\gamma^y(t) = \frac{d\gamma^y(t)}{dt}$.

and let also $\lambda(t) := \lambda(\gamma(t))$, $\frac{\partial \lambda}{\partial x}(t) := \frac{\partial \lambda}{\partial x}(\gamma(t))$ etc.

Then γ is a geodesic if and only if

$$\boxed{\begin{aligned} &\frac{d^2\gamma^x}{dt^2}(t) + \frac{1}{2\lambda(t)} \left[\frac{\partial \lambda}{\partial x}(t) \left(\frac{d\gamma^x}{dt}(t) \right)^2 + 2 \frac{\partial \lambda}{\partial y}(t) \frac{d\gamma^x}{dt}(t) \frac{d\gamma^y}{dt}(t) - \frac{\partial \lambda}{\partial x} \left(\frac{d\gamma^y}{dt}(t) \right)^2 \right] = 0 \\ \text{and} \quad &\frac{d^2\gamma^y}{dt^2}(t) + \frac{1}{2\lambda(t)} \left[\frac{\partial \lambda}{\partial y}(t) \left(\frac{d\gamma^y}{dt}(t) \right)^2 + 2 \frac{\partial \lambda}{\partial x}(t) \frac{d\gamma^x}{dt}(t) \frac{d\gamma^y}{dt}(t) - \frac{\partial \lambda}{\partial y} \left(\frac{d\gamma^x}{dt}(t) \right)^2 \right] = 0 \end{aligned}}$$

(The general formula using Christoffel symbols is not much more complicated, I used $\langle \cdot, \cdot \rangle_p = \lambda_p \text{Eucl.}$ to get rid of Christoffel)

Bottom line: a curve is a geodesic i.f.f. it satisfies some (explicit) system of Ordinary Differential Equations.
One can then appeal to standard analysis theorems to prove the following:

<u>Thm</u> For every $V_p \in T_p\Omega$ there exists an $\varepsilon > 0$ small enough s.t. there is a geodesic $\gamma : (-\varepsilon, \varepsilon) \to \Omega$ s.t. $\gamma(0) = p$ $\dot{\gamma}(0) = V_p$

Furthermore, there exist a <u>unique</u> maximal geodesic $\gamma : I \to \Omega$ where $I \subseteq \mathbb{R}$ is some interval that contains 0, and so that

$\gamma(0) = p$ $\dot{\gamma}(0) = V_p$ (If $I = \mathbb{R}$ we say that γ is a bi-infinite line)

The following is often useful:

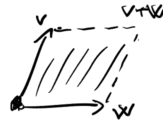
<u>Thm</u> (Hopf–Rinow): The following are equivalent:

1) (Ω, d) is complete (as a metric space)
2) there exists a $p \in \Omega$ such that $\forall V_p \in T_p\Omega$ the maximal geodesic with speed V_p is a bi-infinite line
3) Every geodesic can be extended to a bi-infinite line.

When these conditions hold we further have that for every $p, q \in \Omega$ there exists a geodesic γ connecting them that realizes their distance:

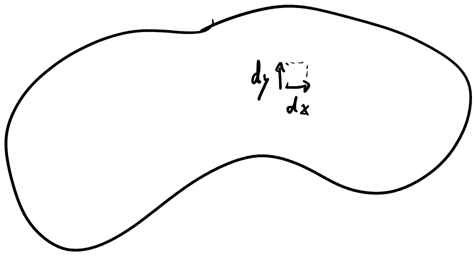
$$d(p,q) = \text{length}(\gamma)$$

§2: Area (Handwavy explaination)



The idea is simple enough: in $\mathbb{E}^2$ the area of a parallelogram is equal to the determinant of the matrix $(v|w)$

Let (Ω, d) be a domain with a Riemannian metric, we want to compute its area.



That should be the integral of the volume of infinitesimal volume elements. I.e. since we have our favourite system of coordinates, it is enough to compute what should be the are of an infinitesimal parallelogram of sides dx and dy

Why do I say 'parallelogram' instead of 'square'? Well, that's because we have to see dx & dy as vectors in $T_p\Omega$ (i.e $\hat{X}_p$ & $\hat{Y}_p$), and $T_p\Omega$ has to be considered with the norm coming from $\langle \cdot, \cdot \rangle_p$

According to this inner product $\hat{X}_p$ & $\hat{Y}_p$ need not be sides of a square.

As usual, let $\langle \cdot, \cdot \rangle_p$ be given by the matrix $\begin{pmatrix} E_p & F_p \\ F_p & G_p \end{pmatrix}$ (in the x,y coord.)

Then $\|\hat{X}\|_p = \sqrt{E_p}$ $\|\hat{Y}_p\| = \sqrt{G_p}$. If $F_p = 0$, then $\hat{X}$ and $\hat{Y}$ are orthogonal, and hence the area of the volume element dx dy is $\sqrt{E_p} \cdot \sqrt{G_p}$

(It is a simple exercise of linear algebra (do it!) that the area (according to $\langle \cdot, \cdot \rangle_p$) of the parallelogram $\hat{X}, \hat{Y}$ is equal to $\sqrt{\det(A_p)}$)

In particular, if $\langle \cdot, \cdot \rangle_p = \lambda(p) \cdot \mathrm{Eucl.}$ then the volume element is $\lambda(p)$

putting things together ⇝

There is a natural measure on (Ω, d).

If $\langle \cdot, \cdot \rangle_p = \lambda(p)\,\mathrm{Eucl.}$, the area of a measurable subset $B \subset \Omega$ is given by
$$\int_\Omega \lambda(p)\, \mathbb{1}_B(p)\ dx\,dy$$

indicator function.

§ 3: Gaussian Curvature (No Riemann tensors)

Unfortunately, I find that the idea behind curvature is sort of mysterious and I cannot explain it without doing a lot of nasty details.

There are a couple of notions of curvature for surfaces

intrinsic: sectional curvature & scalar curvature (scal.curv = 2·sect.curv.)

they are defined only in terms of $\langle\cdot,\cdot\rangle_p$

extrinsic: Gaussian curvature

defined for surfaces embedded into $\mathbb{R}^3$

A version of Gauss's Theorema Egregium is that

Gaussian curvature = sectional curvature (when defined)

The curvature (Gaussian or sectional) at a point p is denoted by $K(p)$

For some unclear reason, I thought it was a good idea to try to find an explicit formula for the curvature $K(p)$ when $\langle\cdot,\cdot\rangle_p = \lambda(p)\,\mathrm{Eucl.}$

Going through [GHL: Def 3.7 (3.A.2) & 3.16 (3.A.3)] and putting together various things, I got:

$$\lambda(p)K(p) = \frac{\partial}{\partial x}\Gamma^{y}_{yx} - \frac{\partial}{\partial y}\Gamma^{y}_{xx} + \underbrace{\sum_{\eta = x,y}\left[\Gamma^{\eta}_{yx}\Gamma^{y}_{x\eta} - \Gamma^{\eta}_{xx}\Gamma^{y}_{y\eta}\right]}$$

All of this cancels out when putting in the values of the Christoffel symbols

$$= \frac{\partial}{\partial x}\left(\frac{1}{2\lambda}\frac{\partial\lambda}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{2\lambda}\frac{\partial\lambda}{\partial y}\right)$$

$$= \frac{1}{2\lambda}\frac{\partial^2\lambda}{\partial x^2} - \frac{1}{2\lambda^2}\left(\frac{\partial\lambda}{\partial x}\right)^2 + \frac{1}{2\lambda}\frac{\partial^2\lambda}{\partial y^2} - \frac{1}{2\lambda^2}\left(\frac{\partial\lambda}{\partial y}\right)^2$$

$$\boxed{K = \frac{1}{2\lambda^2}\left[\frac{\partial^2\lambda}{\partial x^2} - \frac{1}{\lambda}\left(\frac{\partial\lambda}{\partial x}\right)^2 + \frac{\partial^2\lambda}{\partial y^2} - \frac{1}{\lambda}\left(\frac{\partial\lambda}{\partial y}\right)^2\right]}$$

... I'm not sure it was a good idea though

$$= \boxed{-\frac{\Delta(\log\lambda)}{2\lambda}}$$

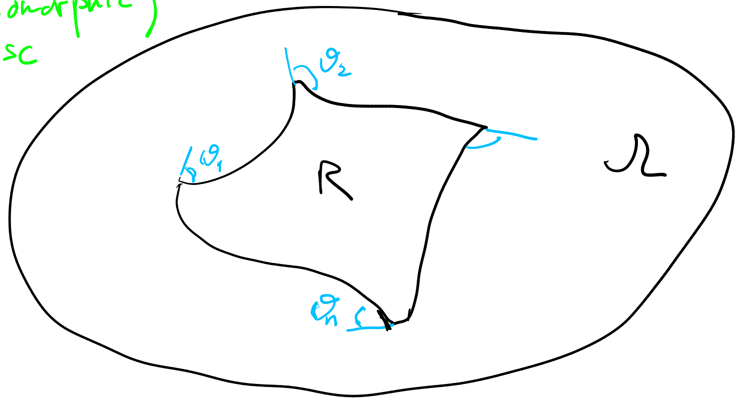
where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian

§4: Gauss-Bonnet

Let R be a simply connected regular domain (i.e. with piecewise smooth boundary) s.t. $\overline{R} \subset \Omega$.

(this is homeomorphic to a disc)

Let $p_1, \dots, p_n$ be the points where ∂R is not smooth, and let $\theta_1, \dots, \theta_n$ be the corresponding external angles



Thm (Gauss-Bonnet (local) [do Carmo p 268]) The following holds:

$$\int_R K(p) \, dArea(p) = 2\pi - \int_{\partial R} k_{\partial R}(s) \, ds - \sum_{i=1}^{n} \theta_i$$

curvature in (Ω, d)

Riemannian Area

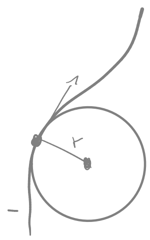
$\boxed{\text{integral along } \partial R \text{ of the "curvature of } \partial R\text{"}}$

sum of external angles

For us, it's enough to know that when ∂R is piecewise geodesic this quantity is 0

(this is something that I did not define. For a curve $\gamma: [0,1] \to \mathbb{R}^2$ its absolute value is given by $|k_\gamma(t)| = \frac{|\gamma''|}{|\gamma'|^2}$

(this is $\frac{1}{(\text{radius of a circle approximating } \gamma)}$)



)

Insofar we only dealt with domains in $\mathbb{R}^2$. For general surfaces, the various notions can be defined in local coordinate charts. Since all the things we used have a "local" definition, they generalize naturally to Riemannian Surfaces. Finally, every (compact) Riemannian surface can be cut into finitely many regions R_i (with piecewise geodetic boundary) where one can apply Gauss Bonnet.

Putting those regions together, it's easy (exercise) to deduce the following

Thm (Gauss Bonnet (Global)) Let (Σ, d) be a compact Riemannian surface with geodesic boundary.

Then
$$\int_\Sigma K \, dArea = 2\pi \, \chi(\Sigma)$$

Euler Characteristic

... This is magic. Appreciate it.