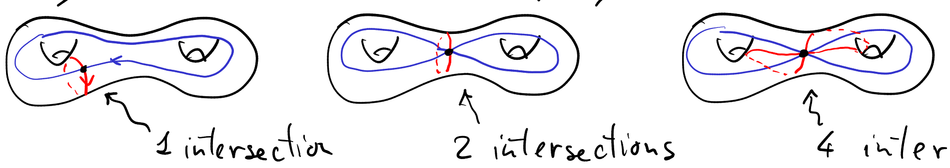
— Exercises Week 9 —

Σ will always denote a compact surface

Def Let $\alpha, \beta : S^1 \to \Sigma$ be two (parametrized) curves in Σ.
Their geometric intersection number $i(\alpha, \beta)$ is the number of times they intersect (with multiplicity)



Informal definition of multiplicity: "wiggle" the curves to make sure that all the intersections are simple and then count those intersections

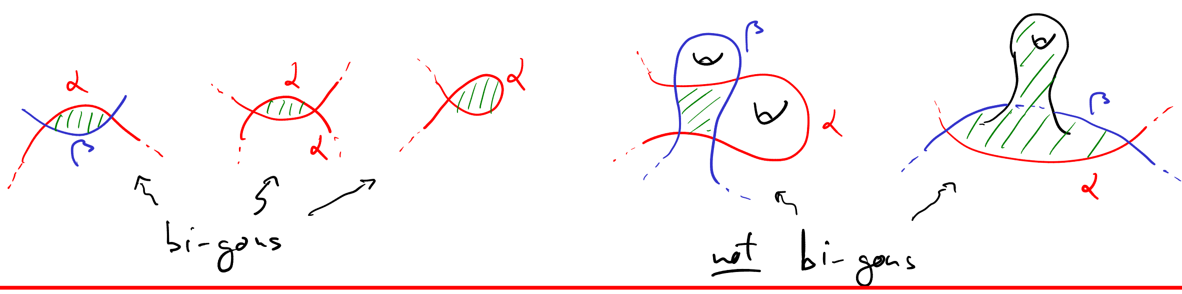
Pedantic definition: let $\alpha \times \beta : S^1 \times S^1 \longrightarrow \Sigma \times \Sigma$ be the product map. the geometric intersection number is $|(\alpha \times \beta)^{-1}(\Delta_\Sigma)|$ (could be infinite)
diagonal

Def two curves α, β are in minimal position if

$$\boxed{i(\alpha,\beta) = \min \{ i(\alpha', \beta') \mid \alpha \sim \alpha' \quad \beta \sim \beta' \}}$$

free homotopy

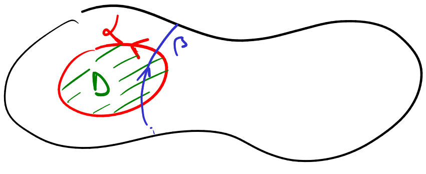
Def Let α, β be two curves. We say that there is a bi-gon between them if $\Sigma \setminus \{\alpha \cup \beta\}$ has a connected component that is homeomorphic to a disc D and such that ∂D consists of (at most) two segments in α and/or β:



Ex1 (The bi-gon criterion)

1) Show that if there is a bi-gon between α and β then they are not in minimal position

2) Let α and β be smooth simple closed curves and assume that α is the boundary of a disk $D \subset \Sigma$ and that β intersects the interior of D.



Prove that there is a bi-gon between α and β

3*) Let α, β be two smooth curves in Σ. Prove that if they are not in minimal position then there is a bi-gon between them
(Hint: use universal covers)

Hard(er)

Ex2) 1) Note that if $\alpha \sim \beta$ and they are in minimal position then $i(\alpha, \beta) = 0$

2) Prove that homotopic multicurves are ambient isotopic

You can assume:
- the bi-gon criterion
- multicurves are ambient-isotopic to smooth multicurves
- homotopically trivial s.c.c bound discs
- disjoint, homotopic s.c.c. bound cylinders (they are parallel)

Ex3) Prove that $\boxed{MCG(\text{Torus}) \cong Sl(2, \mathbb{Z})}$

Ex4) 1) Find 2 simple closed curves α, β in $\Sigma_2 =$  such that $\Sigma_2 \setminus (\alpha \cup \beta)$ is a union of discs

2) Find 2 simple closed curves α, β in Σ_g such that $\Sigma_g \setminus (\alpha \cup \beta)$ is a union of discs

two such curves are said to "fill" the surface