— Exercises Week 8 —

Exercise 1): Let X_1, X_2 be simply connected topological spaces and $\Gamma_1 < \mathrm{Homeo}(X_1)$, $\Gamma_2 < \mathrm{Homeo}(X_2)$ groups of homeo s.t. $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ are free and prop. disc.

a) Show that a homeomorphism $F : X_1/\Gamma_1 \to X_2/\Gamma_1$ must lift to a homeo $\tilde{F} : X_1 \xrightarrow{\sim} X_2$

b) Show that $\boxed{\Gamma_2 = \Gamma_1^{\tilde{F}}} := \{ \tilde{F} \circ \gamma \circ \tilde{F}^{-1} \mid \gamma \in \Gamma_1 \}$

c) Vice versa, show that if $\tilde{F} : X_1 \to X_2$ is a homeo s.t. $\Gamma_2 = \Gamma_1^{\tilde{F}}$ then it descends to a homeo $F : X_1/\Gamma_1 \to X_2/\Gamma_2$

d) prove that the moduli space of a topological surface Σ is equal to

$$\boxed{\left\{ \Gamma < \mathrm{Aut}(X) \;\middle|\; X = \mathbb{S}^2, \mathbb{C}, \mathbb{D},\ X/\Gamma \underset{\text{homeo}}{\cong} \Sigma \right\} \Big/ \sim}$$

where $\Gamma_1 < \mathrm{Aut}(X_1) \sim \Gamma_2 < \mathrm{Aut}(X_2)$ iff. $\exists \tilde{F} : X_1 \xrightarrow{\cong} X_2$ s.t. $\Gamma_2 = \Gamma_1^{\tilde{F}}$

by $\mathrm{Aut}(-)$ I mean "self-conformal equivalences"

Exercise 2) We know that any domain $\Omega \subset \mathbb{C}$ s.t. $\pi_1(\Omega) \cong \mathbb{Z}$ is conformally equivalent to $\mathbb{C}^*$, $\mathbb{D} \setminus \{0\}$ or D_r.

- Show that $\Omega \cong \mathbb{C}^*$ iff $\Omega = \mathbb{C} \setminus \{1 \text{ point}\}$
- Show that $\Omega \cong D_r$ iff $\mathbb{C} \setminus \{0\}$ has 2 connected components, each having more than 1 point.

(Hint: removable singularity)

The next exercise is a different take on the moduli space of the torus.

We say that $\Gamma < \mathbb{C}$ is a lattice if it is discrete and $\Gamma \cong \mathbb{Z}^2$ (so that $\boxed{\mathrm{Mod}.\, S_p(\text{Torus}) = \{\text{lattices}\}/_{\text{conj.}} = \{\text{lattices}\}/_{\mathbb{C}^*}}$)

complex multiplication

Exercise 3) Let $\boxed{\mathrm{Hom}(\mathbb{Z}^2, \mathbb{C})}$ be the group of homomorphisms $\mathbb{Z}^2 \to \mathbb{C}$.
Looking at the images of the generating set $\{(1,0), (0,1)\}$ and identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can see $\varphi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C})$ as a matrix: $(\varphi) \in M_{2\times 2}(\mathbb{R})$ ← here the notation (φ) means that I am looking at φ as a matrix

a) Show that there is a surjection

$$\{\varphi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) \mid \det((\varphi)) > 0\} \twoheadrightarrow \{\text{lattices in } \mathbb{C}\}$$

This naturally descends to a surjection:

$$\boxed{\mathbb{C}^* \backslash \{\varphi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) \mid \det((\varphi)) > 0\} \twoheadrightarrow \{\text{lattices in } \mathbb{C}\}/_{\mathbb{C}^*}}$$

Where I wrote this quotient to the left because we see $\mathbb{C}^* \cong O^+(2, \mathbb{R})$ as acting on $M_{2\times 2}(\mathbb{R})$ by post-composition

$$\underbrace{\lambda e^{i\theta} \cdot \varphi}_{\text{action of } \mathbb{C}^*} = \underbrace{\lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (\varphi)}_{\text{matrix multiplication}}$$

b) Show that $\varphi, \psi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C})$, $\det(\varphi), \det(\psi) > 0$ give the same lattice in $\mathbb{C}$ iff they differ by pre-composition with some $A \in SL(2, \mathbb{Z})$ (i.e. $\varphi = \psi \circ A$)

It follows that

$$\boxed{\begin{aligned} &\{\varphi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) \mid \det((\varphi)) > 0\}/_{SL(2,\mathbb{Z})} \xleftrightarrow{1-1} \{\text{lattices in } \mathbb{C}\} \\ &\text{and} \\ &\mathbb{C}^* \backslash \{\varphi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) \mid \det((\varphi)) > 0\}/_{SL(2,\mathbb{Z})} \xleftrightarrow{1-1} \{\text{lattices in } \mathbb{C}\}/_{\mathbb{C}^*} \end{aligned}}$$

$\boxed{c)}$ Show that there is a bijection

$$\mathbb{C}^* \backslash \{ \varphi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) \mid \det((\varphi)) > 0 \} \overset{1-1}{\longleftrightarrow} \text{Half Plane}$$

(wlog $(1,0)$ is sent to $1 \in \mathbb{C}$ ...)

$\boxed{d)}$ Show that the action

$$\mathbb{C}^* \backslash \{ \varphi \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) \mid \det((\varphi)) > 0 \} \circlearrowleft SL(2, \mathbb{Z})$$

is by Möbius transformations

(Warning $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will not act as $z \mapsto \frac{az+b}{cz+d}$

This is not worrying: after all, we are dealing with a right-action, not the natural left-action)

$\boxed{e)}$ Find a fundamental domain for this action, and conclude that it is a model for the moduli space of the torus

Exercise 4) Show that if Σ is a closed (cpt, no boundary) topological surface then there is a natural 1-1 correspondence

$$\boxed{\mathrm{ModSp}(\Sigma) \longleftrightarrow \{ \text{metrics of constant curvature} \} / \text{isometries and rescaling}}$$

Recall:
- Constant positive $\longleftrightarrow \lambda \cdot \mathbb{S}^2$ (sphere of rad. λ)
- Flat $\longleftrightarrow$ Euclidean
- Constant negative $\longleftrightarrow \lambda \cdot \mathbb{D}$ (rescaled Poincaré disk)