— Exercises Week 6 —

Let $p,q,r,s \in \mathbb{C}$ be distinct points, the cross-ratio $[p,q,r,s]$ is defined as:

$$\boxed{[p,q,r,s] := \frac{(p-q)(r-s)}{(p-s)(q-r)}} \in \mathbb{C}$$

Exercise 1 a) Show that the map $z \xmapsto{\phi} [z,q,r,s]$ is a Möbius Transf. that sends q,r,s to $0,1,\infty$

b) Show that for every 2 triples $\{q,r,s\},\{q',r',s'\}$ of distinct points, and $z \notin \{q',r',s'\}$ the equation $[w,p,q,r] = [z,p',q',r']$ has a unique solution $w(z)$.
The map $w \longmapsto w(z)$ is a Möbius Transformation.

c) Show that the cross ratio is invariant under Möbius Transformation

$$\boxed{[p,q,r,s] = [\phi(p),\phi(q),\phi(r),\phi(s)]}$$

$\forall \phi$ Möbius Transf.

d) Deduce that for every 2 triples of distinct points $\{q,r,s\},\{q',r',s'\}$ there exist a unique Möbius Transf. ϕ s.t. $\phi(q)=q'$ $\phi(r)=r'$ $\phi(s)=s'$

Optional: Show that $[p,q,r,s]$ is real iff. p,q,r,s lie on a line or a circle.
Show that $[p,q,r,s] = \overline{[\tilde{p},q,r,s]}$ (complex conjugate) iff $\tilde{p}$ is obtained as the reflection of p across the unique line/circle passing through q,r,s

Exercise 2) Complete the proof that $\boxed{\mathrm{Aut}(S^2) \cong \mathbb{P}SL(2,\mathbb{C})}$

[the proof is sketched in Chapter 3, §2 (p. 9)]

Exercise 3) Show that $\boxed{\mathrm{Stab}_{\text{half plane}}(\mathbb{P}SL(2,\mathbb{C})) = \mathbb{P}SL(2,\mathbb{R})} < \mathbb{P}SL(2,\mathbb{C})$

$M_A(z) = \dfrac{za+b}{zc+d}$ Möbius transformation

Exercise 4) Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $t_A = \mathrm{tr}(A)$ be the trace.

We know that M_A is parabolic iff $\boxed{t_A^2 = 4.}$

Show that
- M_A is elliptic $\iff$ $\boxed{t_A^2 \in [0,4)}$
- " " hyperbolic $\iff$ $\boxed{t_A^2 \in (4,\infty)}$ (i.e. is real and >4)
- " " general loxodromic $\iff$ $\boxed{t_A^2 \in \mathbb{C} \setminus [0,\infty)}$