— Exercises, Week 5 —

The first exercise is completing a proof that we sketched in class:

Ex1) Show in some detail that

a) if Σ is a Riemann surface and $\pi : \tilde{\Sigma} \to \Sigma$ is a covering (not necessarily the universal cover), then $\tilde{\Sigma}$ admits a complex structure such that π is holomorphic and $\mathrm{Aut}(\tilde{\Sigma} \to \Sigma)$ acts by bi-holomorphisms.
(We say that π is a holomorphic covering

b) Conversely, show that if $\Gamma \curvearrowright \Sigma$ is a free, prop. disc. action by bi-holomorphisms on a Riemann surface then the quotient Σ/Γ has a complex structure such that $\pi : \Sigma \to \Sigma/\Gamma$ is holomorphic

Ex2) Show that Exercise 1 holds when replacing "Complex/Riemann" with "Riemannian" and "holomorphic" with "Riemannian-isometric"

precisely, the quotient $\pi : \tilde{\Sigma} \to \Sigma$ will be a local isometry, while the elements in $\mathrm{Aut}(\tilde{\Sigma} \to \Sigma)$ will be global isometries

Ex3) Let X be a path-connected topological space and $G := \mathrm{Homeo}(X)$ be the group of self-homeo.
Let $\Gamma < G$ be a subgroup such that $\Gamma \curvearrowright X$ is free and properly disc.
and let $N_G(\Gamma) < G$ be the normalizer of Γ in G.

a) show that there is a homomorphism $\Psi : N_G(\Gamma) \longrightarrow \mathrm{Homeo}(X/\Gamma)$

b) show that Ψ need not be surjective in general

c) show that if X is simply-connected then Ψ is surjective and descends to an isomorphism $N_G(\Gamma)/\Gamma \cong \mathrm{Homeo}(X/\Gamma)$

d) Note that if $X = \Sigma$ is a Riemann or Riemannian surface then the statements remain true if one replaces "homeo" by "biholom." or "isometry"