— Exercises Week 4 —

Recall: $\Sigma_{g,b}$ = surface of genus g and b boundary components

Exercise 1) Show that $\chi(\Sigma_{g,b}) = 2-2g-b$ ← Euler characteristic

Deduce that if $\Sigma_{g,b}$ and $\Sigma_{g',b'}$ are homeomorphic then $g=g'$ and $b=b'$

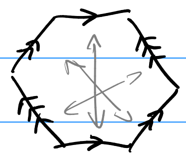
(this is one half of the classification of surfaces: we now know that the $\Sigma_{g,b}$ are all different. Once we know that every cpt. surface is homeom. to one of these, we get a complete classification)

Exercise 2)

- Prove that $\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1] \cdots [a_g,b_g] \rangle$
- Prove that $\pi_1(\Sigma_{g,b}) = F_{2g+b-1}$ if $b \geq 1$

free group in $2g+b-1$ generators

Exercise 3) Show that $\Sigma_g = (4g+2)$-gon where opposite edges are identified

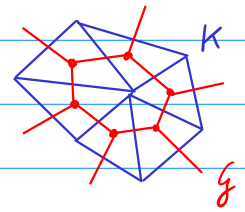


thank you to Hanan for pointing out this proof to me. It is apparently due to [Zeeman]

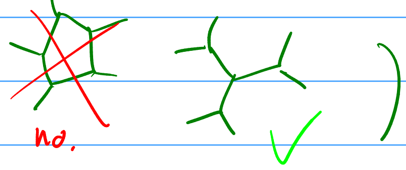
Exercise 4 (Classification of closed surfaces)

Let $\Sigma = |K|$ be a compact triangulated surface with no boundary.
Let $\mathcal{G}$ be the dual graph of K

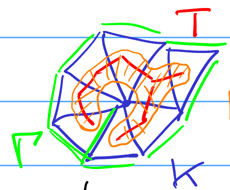
(vertices of $\mathcal{G}$ $\longleftrightarrow$ triangles of K
edges of $\mathcal{G}$ $\longleftrightarrow$ crossing over edges of K)



Let $T \subset \mathcal{G}$ be a maximal tree
(i.e. a subgraph that has no closed loops
and such that any larger graph has closed loops)



Step 1: Show that every vertex in $\mathcal{G}$ belongs to T and that a neighbourhood of T is homeomorphic to a disk



Let $\Gamma < K$ be the graph consisting of those edges in K that don't intersect T.
The Euler characteristic of a graph is #vertices − #edges

Step 2 Note that $\chi(\Sigma) = \chi(T) + \chi(\Gamma)$
deduce that $\chi(\Sigma) \leq 2$

Step 3 $\chi(\Sigma) = 2$ iff Γ is a tree. In this case Σ is homeomorphic to a sphere $\mathbb{S}^2$.

Step 4 if Γ is not a tree, then Σ has a non-separating simple closed curve

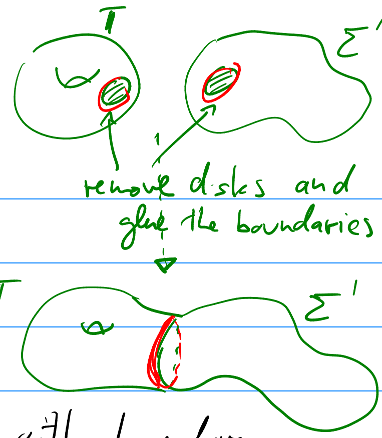
$\Sigma \setminus \gamma$ is connected

$\gamma : S^1 \to \Sigma$ is injective

Step 5 if Σ has a non-separating simple closed curve then
$\Sigma = \mathbb{T} \# \Sigma'$ where Σ' is a closed surface with
$\chi(\Sigma') \gneq \chi(\Sigma)$

torus connected sum

(i.e. $\Sigma =$



)

Step 6 complete the proof by induction on $\chi(\Sigma)$

Optional step: prove the classification for surfaces with boundary

Recall that a Riemannian metric on a domain Ω induces a notion of Volume and of curvature $K : \Omega \to \mathbb{R}$. We did not define them formally, but if we did it would be clear that these are local notions (i.e. volume and curvature at a point $p \in \Omega$ are determined by the restriction of the Riemannian metric to any small neighborhood $p \in U \subseteq \Omega$). Further, they are invariant under Riemannian isometries.

Bottom line: By pasting local charts, we get a well defined notion of Volume and curvature $K : \Sigma \to \mathbb{R}$ on every Riemannian surface (Σ, d).
In particular, we can consider $\int_\Sigma K(p)\, dArea$.

Optional:

Ex 5* (Gauss-Bonnet for surfaces) Let Σ be a cpt Riemannian Surface with geodesic boundary. Prove that

$$\boxed{\int_\Sigma K(p)\, dArea = 2\pi\, \chi(\Sigma)}$$

(Assume the Gauss-Bonnet formula for polygons in planar domains. Solving this exercise in detail is going to take time)