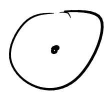
Exercises – Week 3

Ex 1 Let R_1 and R_2 be two rectangles in $\mathbb{C}$ 
Assume that there exists a conformal equivalence $F: R_1 \to R_2$ such that F sends the vertices of R_1 to the vertices of R_2 (recall that Carathéodory's theorem implies that $\exists F$).

Prove that there exist $\tilde{F}: \mathbb{C} \to \mathbb{C}$ bi-holomorphic and such that $\tilde{F}|_{R_1} = F$.

(Hint: Reflection principle)

Rmk with a bit of extra work, one can use Ex 1 to show that such an F must be multiplication by a complex number.

Ex 2 (Weierstrass theorem) Let $\Omega = \{ z \mid 0 < |z| < 1 \}$ 
and let $F: \Omega \to \mathbb{C}$ be holomorphic and such that $F(\Omega)$ is not dense in $\mathbb{C}$.

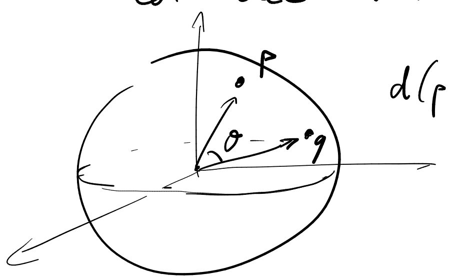
a) Let z_0 be in $\mathbb{C} \setminus \overline{F(\Omega)}$. Show that $G(z) := \dfrac{1}{F(z) - z_0}$ is holomorphic on Ω and it is bounded.

It follows that G extends to a holomorphic map $\tilde{G}: \mathbb{D} \to \mathbb{C}$ (Riemann extension Theorem)

b) Prove that there is an n large enough so that $z \mapsto z^n \cdot F(z)$ is bounded on $\{ z \mid 0 < |z| < \frac{1}{2} \}$ (and hence extends to a holomorphic map on $\mathbb{D}$)

Ex 3 a) Show that the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ can be equipped with a complex structure (i.e. it has an atlas with bi-holomorphic change-of-coordinates)

b) Show that $\mathbb{S}^2$ has a Riemannian metric such that the induced metric coincides with the angular distance



$d(p,q) = \vartheta$

The following exercises are a guided proof of the Riemann Mapping Theorem.

Fact/Ex 4.0 If $\Omega \subset \mathbb{C}$ is simply connected and $0 \notin \Omega$ then there exists $\sigma : \Omega \to \mathbb{C}$ holomorphic and such that $\sigma(z)^2 = z \quad \forall z \in \Omega$
(that is σ is a holomorphic branch of the square root)

(You can prove it using path-integrals and the fact that $\int_\gamma F = 0$ if γ is closed & F holomorphic. See Exercise 3 of last week.)

Ex 4.1 (reduction to bounded domains) Let $\Omega \subsetneq \mathbb{C}$ be a proper simply-connected domain, and let $z_0 \in \mathbb{C} \setminus \Omega$.
Consider $z \mapsto \sigma(z - z_0)$ (σ as above).
Show that there exists a $w \in \mathbb{C}$ and $r > 0$ s.t.

$$B(w, r) \cap \varphi(\Omega) = \emptyset.$$

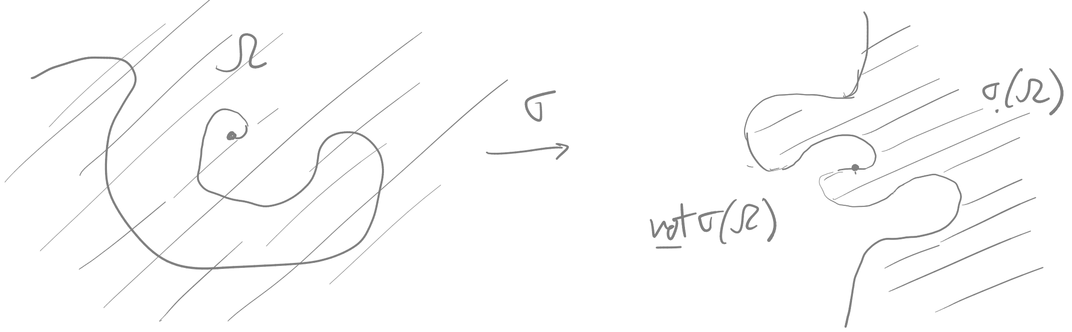
Deduce that there exists an injective holomorphic map

$$F : \Omega \to \mathbb{D}$$

in particular Ω is conf. eq. to $F(\Omega)$

(Hint: σ holomorphic $\Rightarrow$ open. Moreover $-\sigma(z) \notin \sigma(\Omega)$ if $z \in \Omega$)

Picture:



Given $\alpha \in \mathbb{C}$, let $\varphi_\alpha(z) := \dfrac{z - \alpha}{1 - \bar{\alpha} z}$ ← conjugate

Ex 4.2 Show that for every $\alpha \in \mathbb{D}$ $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ is a bi-holomorphism such that $\varphi_\alpha(\alpha) = 0$.

Deduce that it is enough to prove the Riemann Mapping Thm for domains $\Omega \subset \mathbb{D}$ such that $0 \in \Omega$

Fact (Schwarz Lemma) let $F : D \to D$ be holomorphic. such that $F(0) = 0$. Then

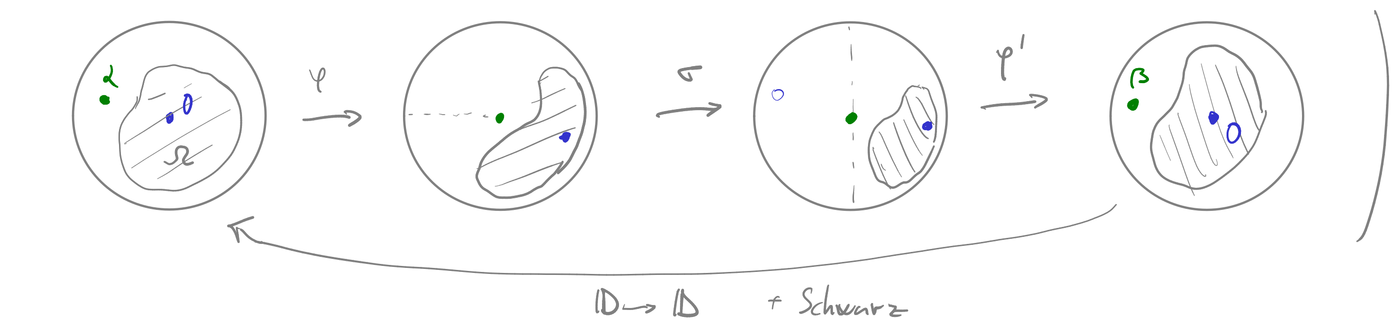
- $|F(z)| \leq |z| \quad \forall z \in D$
- $|F'(0)| \leq 1$

Moreover, if $|F(z)| = |z|$ for some $z \neq 0$ or $|F'(0)| = 1$ then $\exists \lambda \in \mathbb{C}$ with $|\lambda| = 1$ s.t. $F(z) = \lambda z$ (i.e. F is a rotation of the disk)

Ex 4.3 Show that if we are given $0 \in \Omega \subset \mathbb{D}$ such that $\Omega \neq \mathbb{D}$, then there exists an injective holomorphic function $F : \Omega \to \mathbb{D}$ such that $F(0) = 0$ and $|F'(0)| > 1$.

Deduce that if $\mathcal{F} := \{ F : \Omega \to \mathbb{D} \mid F \text{ injective holomorphic}, F(0) = 0 \}$ and $\tilde{F} \in \mathcal{F}$ is such that $|\tilde{F}'(0)| = \max_{F \in \mathcal{F}} |F'(0)|$, then $\tilde{F} : \Omega \xrightarrow{\cong} \mathbb{D}$ is a conformal equivalence

Hint:



To conclude the proof of the theorem we thus need to show that $\exists \hat{F} \in \mathcal{F}$ that maximizes $|F'(0)|$. We use the following:

Fact Every sequence $F_n \in \mathcal{F}$ admits a subsequence F_{n_k} that converges uniformly on compact sets to a holomorphic function $\hat{F} : \Omega \to \mathbb{D}$

(I.e. $\mathcal{F}$ is a Normal Family. The proof is not hard, you can read it e.g. on [Rudin, Thm 14.6])

Choose a sequence $F_n \in \mathcal{F}$ such that $|F_n'(0)| \to$ max derivative.
WLOG $F_n \xrightarrow[n\to\infty]{} \hat{F}$ uniformly on compact sets.

$\underset{\text{analysis}}{\rightsquigarrow}$ $\hat{F}'(0) = \lim_{n\to\infty} F_n'(0)$ = max derivative.

If we can show that $\hat{F} \in \mathcal{F}$ we're done.

- $\hat{F}(0) = 0$ ✓
- $\hat{F}(\Omega) \subseteq \overline{\mathbb{D}}$. Since holomorphic maps are open, we get $\hat{F}(\Omega) \subseteq \mathbb{D}$ ✓

Remains to check:

Ex 4.4 ($\hat{F}$ is injective):

$$\begin{array}{ccc} F_n(w) & \longrightarrow & \hat{F}(w) \\ \| & & \| \\ w_n & \longrightarrow & \hat{w} \end{array}$$

Step 1: Fix $v \neq w \in \Omega$. then
Show that there is a neighbour $U = B(v, r)$ such that
- $w \notin U$
- the function $z \mapsto \hat{F}(z) - \hat{w}$ has no zeros on ∂U.

Step 2 Note that the functions $F_n - w_n$ converge to $\hat{F} - \hat{w}$ and have no zeros in U

Step 3 Google Rouché's Theorem and use it on ∂U to deduce that $\hat{F}(v) \neq \hat{F}(w)$