— Exercise Sheet 2 —

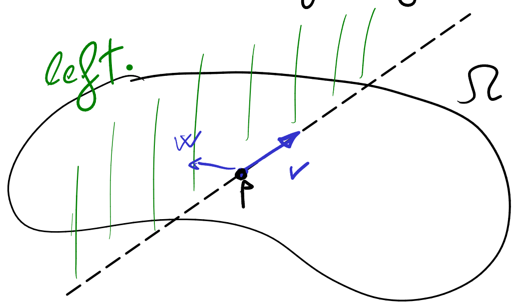
Def a differentiable map $F: \Omega \to \mathbb{R}^2$ is orientation preserving if $d_pF : T_p\Omega \to T_{F_p}\Omega$ has determinant
$T_p\Omega \cong \mathbb{R}^2$, $T_{F_p}\Omega \cong \mathbb{R}^2$

$$\det(d_pF) > 0 \quad \forall p \in \Omega$$

Def the absolute value of the Riemannian angle between $v, w \in T_p\Omega$ is

$$|\sphericalangle(v,w)| := \cos^{-1}\left(\frac{\langle v,w\rangle_p}{\|v\|_p \|w\|_p}\right) \in [0,\pi]$$

The sign of the Riemannian angle is positive if w belongs to the half-space to the left of v:



Ex 1) Prove in detail that a differentiable map $F : (\Omega_1, d_1) \to (\Omega_2, d_2)$ s.t. d_pF is injective $\forall p \in \Omega_1$ preserves the absolute value of the angles (with respect to the Riemannian metrics) if and only if $F^*\langle \cdot,\cdot\rangle^{\Omega_2} = \lambda(p)\langle\cdot,\cdot\rangle^{\Omega_1}$.

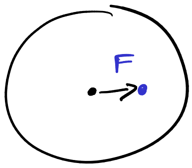
Show that the sign of the angle is preserved iff F is orientation-preserving

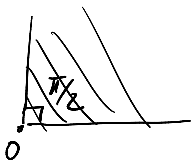
Recall: a conformal diffeomorphism is a diffeomorphism $F:\Omega_1 \to \Omega_2$ that preserves the (Euclidean) angles with sign.

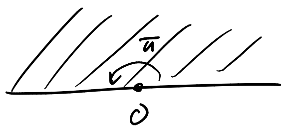
Ex 2 Find a conformal diffeo. $F:\Omega_1 \to \Omega_2$ where:

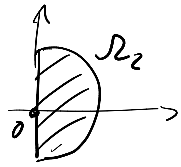
a) $\Omega_1 = D_1$ $\qquad\qquad$ $\Omega_2 =$ Upper half-plane

b) $\Omega_1 = \Omega_2$ $\qquad$ and $\qquad$ $F(0) = \frac{1}{2}$

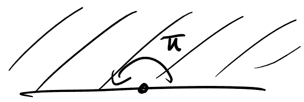


c) $\Omega_1 =$ 

$\qquad$ $\Omega_2 =$ 

d) $\Omega_1 = D_1$ $\qquad\qquad$ $\Omega_2 = D_1 \cap \{Re(z) > 0\}$ 

e) $\Omega_1 =$ 

$\quad \theta \in (0, \pi]$ $\qquad$ $\Omega_2 =$ 

READ ME

The next exercises will prove Cauchy's theorem. They are only meant for those of you who are not confident with complex analysis and would like to remember what goes into the proof of Cauchy's. You can safely skip exercise 3.0, 3.2. The others are important for the proof.

Recall $F:\Omega \to \mathbb{C}$ is holomorphic if its differential d_pF is $\mathbb{C}$-linear $\forall p \in \Omega$

i.e. rotation + dilatation,

In this case we denote the differential d_pF by $F'(p) \in \mathbb{C}$

complex derivative

Ex 3.0 : Show that :

- if $F_1, F_2 : \Omega \to \mathbb{C}$ are holomorphic then $(F_1 \cdot F_2) : \Omega \to \mathbb{C}$ is holomorphic and
$$(F_1F_2)' = F_1'F_2 + F_1F_2'$$
Deduce that $\begin{array}{l} F:\mathbb{C} \to \mathbb{C} \\ \quad z \mapsto z^n \end{array}$ is holomorphic and $F'(z) = nz^{n-1}$

- $\begin{array}{l} F: \mathbb{C}\setminus\{0\} \to \mathbb{C} \\ \qquad z \mapsto \frac{1}{z} \end{array}$ is holomorphic

- if $F_1 : \Omega_1 \to \Omega_2$ and $F_2 : \Omega_2 \to \mathbb{C}$ are holomorphic then $F_2 \circ F_1 : \Omega_1 \to \mathbb{C}$ is holomorphic and
$$(F_2 \circ F_1)'(z) = F_2'(F_1(z))\, F_1'(z)$$

- if $F : \Omega_1 \to \Omega_2$ is a diffeomorphism and it is holomorphic, then $F^{-1} : \Omega_2 \to \Omega_1$ is holomorphic as far as we are concerned, this is just a convenient way to write the point of coordinates (x,y)

Ex 3.1 given $F : \Omega \to \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$) write $F(x+iy) := f(x+iy) + i\, g(x+iy)$
where $f, g : \Omega \to \mathbb{R}$ are real-valued functions.
Show that F is holomorphic if and only if

$$\boxed{\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial g}{\partial x} = -\frac{\partial f}{\partial y}}$$

These are called Cauchy-Riemann equations

Furthermore, we then have

$$F'(x+iy) = \frac{\partial f}{\partial x} + i\frac{\partial g}{\partial x}$$

Warning: this is not the way that one usually thinks about complex derivatives when doing complex analysis

Def let $\gamma : [a,b] \to \Omega$ be a smooth path $a < b \in \mathbb{R}$
Define the path integral:

These are complex!

$$\int_\gamma F(z)\,dz := \int_a^b F(\gamma(t)) \left(\frac{d\gamma_x}{dt}(t) + i \frac{d\gamma_y}{dt}(t) \right) dt$$

the integral of a complex function f is $\int \mathrm{Re}(f) + i \int \mathrm{Im}(f)$

$\gamma(t) = (\gamma_x(t), \gamma_y(t))$

$\frac{d\gamma_x}{dt}$ and $\frac{d\gamma_y}{dt}$ are real numbers

Ex 3.2) Show that if $\psi : [a,b] \to [a',b']$ is a diffeomorphism s.t. $\psi(a) = a'$ $\psi(b) = b'$ and $\gamma : [0,1] \to \Omega$ is a smooth path, then

$$\int_\gamma F(z)\,dz = \int_{\gamma \circ \psi} F(z)\,dz$$

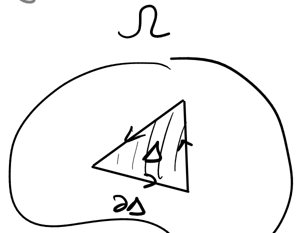
Rmk if γ is a piecewise smooth path, then the integral $\int_\gamma F(z)\,dz$ is defined as the sum of the integral of the smooth pieces

<u>Exercise 3.3</u> Show that for any smooth curve $\gamma : [0,1] \to \Omega$ and holomorphic function $F : \Omega \to \mathbb{C}$ we have

$$\int_\gamma F'(z)\,dz = F(1) - F(0)$$

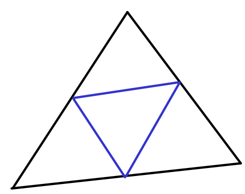
<u>Rmk</u>: a priori there's no reason why this should be true: we defined integration along paths in a fairly arbitrary way. This result tells us that our choices were good ones.

<u>Ex 3.4</u> Assume that $F : \Omega \to \mathbb{C}$ is holomorphic and $\Delta \subset \Omega$ is a compact triangle.



$\partial\Delta$ can be seen as a piecewise smooth path. The aim of this exercise is to show that $\int_{\partial\Delta} F(z)\,dz = 0$

<u>Step 1</u>: Note that $\int_{\partial\Delta} F(z)\,dz$ is equal to the sum of the integrals over boundaries of 4 triangles with halved edge-lengths



<u>Step 2</u> deduce that there exists a sequence of nested triangles $\Delta_n \subset \Delta_{n+1} \subset \cdots$ such that

$$\left| \int_{\partial\Delta} F(z)\,dz \right| \leq 4^n \left| \int_{\partial\Delta_n} F(z)\,dz \right|$$

and diameter of Δ_n is $\operatorname{diam}(\Delta_n) = \frac{1}{2^n} \operatorname{diam}(\Delta)$.

By compactness, it follows that there exist a $z_0 \in \Omega$ such that $z_0 \in \Delta_n \quad \forall n$.

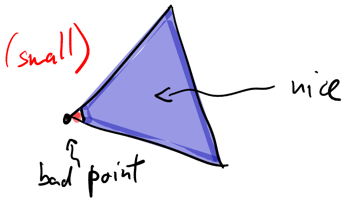
<u>Step 4</u> show that $$\int_{\partial\Delta_n} F(z)\,dz = \int_{\partial\Delta_n} \left[F(z) - F(z_0) - F'(z_0)(z - z_0) \right] dz$$

<u>Step 3</u> use the definition of differential to show that

$$\left| \int_{\partial\Delta} F(z)\,dz \right| < \varepsilon \quad \forall \varepsilon > 0.$$

$\square$

Ex 3.5 Show that **Ex 3.6** remains true if $F : \Omega \to \mathbb{C}$ is continuous and holomorphic on $\Omega \setminus \{p\}$ for a point $p \in \Omega$

(Hint : 

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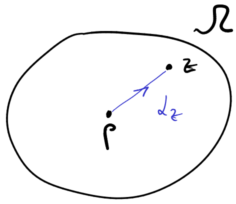
Ex 3.6 (Cauchy) Let Ω be convex and $F : \Omega \to \mathbb{C}$ be continuous and holomorphic on $\Omega \setminus \{p\}$.
Then $\exists\, G : \Omega \to \mathbb{C}$ holomorphic s.t. $F(z) = G'(z)$.
In particular, $\int_\gamma F(z)\,dz = 0$ for every closed path contained in Ω

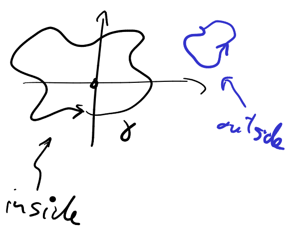
(Hint : 

consider $\int_{d_z} F(z)\,dz$)

Ex 3.7 Show that for $\gamma : [0, 2\pi] \to \mathbb{C}$, $\theta \mapsto e^{i\theta}$ we have $\int_\gamma \frac{1}{z}\,dz = 2\pi i$

Deduce that for every smooth injective map $\gamma : S^1 \to \mathbb{C} \setminus \{0\}$

$$\int_\gamma \frac{1}{z}\,dz = \begin{cases} \pm 2\pi i & \text{if } 0 \text{ is "inside"} \\ 0 & \text{if } 0 \text{ is "outside"} \end{cases}$$



At this point it doesn't take too long to show that holomorphic functions are analytic (i.e. can be locally expressed as power-series). Yet, this problem sheet is alredy too long, so I'll stop it here