Exercises : Week 11

Ex 1) Let Σ be a surface of genus $g \geq 2$.
Show that $MCG(\Sigma) \curvearrowright \mathrm{Teich}(\Sigma)$ and that

$$\boxed{\mathrm{Teich}(\Sigma)/MCG(\Sigma) = \text{Moduli Space}(\Sigma)}$$

Ex 2) Let $\mathbb{T}$ be the torus. We define its Teichmüller space as the set of marked Euclidean structures up to equivalence

$$\boxed{\mathrm{Teich}(\mathbb{T}) := \left\{ \phi : \mathbb{T} \to (\mathbb{T}, d) \;\middle|\; \widetilde{\mathbb{T}} \underset{\text{isom}}{\cong} \mathbb{R}^2 \right\} / \sim}$$

a) Show that there is a natural bijection $\boxed{\mathrm{Teich}(\mathbb{T}) \longleftrightarrow \text{Upper Half Plane}}$ $= \mathbb{R} \times \mathbb{R}_{>0}$

b) How does $MCG(\mathbb{T})$ act on the Upper Half Plane?

Remember that $\boxed{\mathrm{Isom}^+(\mathbb{H}^2) \cong \underbrace{\mathrm{Aut}}_{\text{conformal equivalences}}(\mathbb{D}) \cong PSL(2,\mathbb{R})}$.
In Exercise 1 of Week 8 you had to show that the moduli space of a (closed) topological surface Σ of genus $g \geq 2$ is naturally equal to

$$\boxed{\left\{ \Gamma < PSL(2,\mathbb{R}) \;\middle|\; \mathbb{H}^2/\Gamma \underset{\text{homeo}}{\cong} \Sigma \right\} / \sim}$$

where $\Gamma_1 \sim \Gamma_2$ iff $\exists F \in PSL(2,\mathbb{R})$ such that $\Gamma_1 = \Gamma_2^F = \{ F \circ \gamma \circ F^{-1} \mid \gamma \in \Gamma_2 \}$.

The following exercise is an analogue of this fact for the Teichmüller space:

Ex 3) Let Σ be a surface of genus ≥ 2

a) Show that every hyperbolic marking $\phi : \Sigma \to (\Sigma, d)$ gives rise to a (non-canonical) injective homomorphism

$$\boxed{\pi_1(\Sigma) \longrightarrow PSL(2,\mathbb{R})}$$

whose image is discrete in $PSL(2,\mathbb{R})$

b) Let $\psi : \pi_1(\Sigma) \longrightarrow PSL(2,\mathbb{R})$ be an injective homomorphism with discrete image. Show that $\mathbb{H}^2 / \psi(\pi_1(\Sigma))$ is homeomorphic to Σ (Hint: show that the quotient is a surface and use the classification)

c) Show that there is a natural correspondence

$$\boxed{Teich(\Sigma) \xleftrightarrow{1-1} \left\{ \psi : \pi_1(\Sigma) \to PSL(2,\mathbb{R}) \;\middle|\; \begin{array}{l} \text{injective homomorphism} \\ \text{with discrete image} \end{array} \right\} \Big/ PSL(2,\mathbb{R})}$$

where $PSL(2,\mathbb{R})$ acts by conjugation on the set of homomorphisms:

$$F \cdot \psi(g) := F \circ \psi(g) \circ F^{-1} \qquad \forall F \in PSL(2,\mathbb{R}) \cong Isom^+(\mathbb{H}^2), \; g \in \pi_1(\Sigma)$$

Note that part (b) implies that

$$\boxed{\text{Moduli Space}(\Sigma) = \left\{ \Gamma < PSL(2,\mathbb{R}) \;\middle|\; \Gamma \text{ discrete}, \Gamma \cong \pi_1(\Sigma) \right\} \Big/ \sim}$$

From this point of view, the difference between Teichmüller and Moduli space is that the latter only looks at subgroups of $PSL(2,\mathbb{R})$ that are "abstractly" isomorphic to $\pi_1(\Sigma)$, while the former cares about subgroups together with a fixed, concrete isomorphism with $\pi_1(\Sigma)$.