— Exercises Week 10 —

Again, all surfaces are assumed to have genus ≥ 2

Exercise 1 Let $g \geq 2$, Show in detail that there is a natural bijection

$$\boxed{\text{Moduli space}(\Sigma_g) \longleftrightarrow \{\text{hyperbolic metrics on } \Sigma_g\}/\text{isometry}}$$

{conformal structures}/conf. eq.

Exercise 2 Let $F: \mathbb{H}^2 \to \mathbb{H}^2$ be an isometry and let $\boxed{\delta(F) := \inf\{d(x, F(x)) \mid x \in \mathbb{H}^2\}}$ be the minimum displacement of F.

a) Show that F is

- hyperbolic if $\delta(F) > 0$ (in which case $\exists x$ s.t. $d(x, F(x)) = \delta(F)$)
- parabolic if $\delta(F) = 0$ but $\nexists x \in \mathbb{H}^2$ $d(x, F(x)) = 0$
- elliptic if $\delta(F) = 0$ and $\exists x \in \mathbb{H}^2$ $d(x, F(x)) = 0$

b) Assume that F is hyperbolic. Show that the set $\boxed{\{x \mid d(x, F(x)) = \delta(F)\}}$ is a geodesic in $\mathbb{H}^2$

this set is the axis of F

Let (Σ, d) be a hyperbolic closed surface. Fixing base points defines a (non-canonical) isomorphism $\Psi: \pi_1(\Sigma, x) \xrightarrow{\cong} \mathrm{Aut}(\mathbb{H}^2 \to \Sigma)$

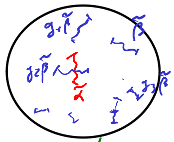
c) Given $[\alpha] \in \pi_1(\Sigma, x)$, show that the unique geodesic that is (freely) homotopic to α coincides with the projection of the axis of $\Psi([\alpha])$. Furthermore, the length of this geodesic is $\delta(\Psi([\alpha]))$.

d) Let γ be the geodesic homotopic to α. Show that $p^{-1}(\gamma)$ is the union of the set of axis of the conjugates of $[\alpha]$

$$p^{-1}(\gamma) = \bigcup_{[\beta] \in \pi_1} \text{Axis of}\left(\Psi([\beta\alpha\beta^{-1}])\right)$$

(Note that the axis of $G \circ F \circ G^{-1}$ is $G(\text{axis of } F)$)

Exercise 3 Let $\alpha, \beta : [0,1) \to \Sigma$ be two curves. Let $\tilde{\alpha}, \tilde{\beta} : [0,1) \to \mathbb{H}^2$ be lifts to the universal cover.



a) prove that $\boxed{i(\alpha,\beta) = \sum_{g \in \mathrm{Aut}(\mathbb{H}^2 \to \Sigma)} |g(\tilde{\beta}) \cap \tilde{\alpha}|}$

intersection number (with multiplicity, see last week's exercises)

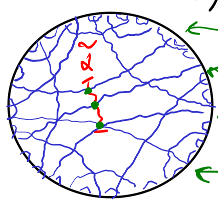
We say that a closed curve is <u>primitive</u> if it is not homotopic to a power of a curve:

primitive   not primitive
(it's like going twice around this)

b) Let β be a primitive closed curve (not necessarily simple) and $\alpha : [0,1) \to \Sigma$ as above (could be closed or not). Then

$$\boxed{i(\alpha,\beta) = \#\left\{\begin{array}{l}\text{bi-infinite lifts } \tilde{\beta} : \mathbb{R} \to \mathbb{H}^2 \\ \text{such that } \tilde{\beta} \cap \tilde{\alpha} \neq \emptyset\end{array}\right\}}$$

$\tilde{\alpha} : [0,1) \to \mathbb{H}^2$ finite lift



c) Show that if $\alpha, \beta : S^1 \to \Sigma$ are <u>distinct geodesic</u> then they are in <u>minimal position</u>.

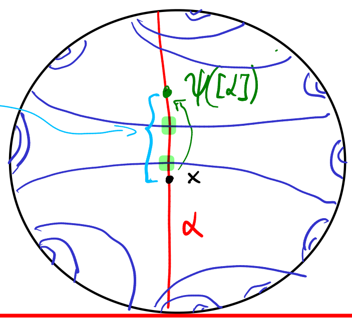
Given two closed curves $\alpha, \beta : S^1 \to \Sigma$, let

$$\boxed{i([\alpha],[\beta]) := \min \left\{ i(\alpha', \beta') \;\middle|\; \begin{array}{l}\alpha \sim \alpha' \\ \beta \sim \beta'\end{array} \text{ (free) homotopy} \right\}}$$

Cor If we are given two curves α, β then $\boxed{i([\alpha],[\beta]) = i(\bar{\alpha}, \bar{\beta})}$ where $\bar{\alpha}, \bar{\beta}$ are the geodesic in $[\alpha], [\beta]$.

Combining Ex 2 and 3, we see that when β is primitive $i([\alpha],[\beta])$ is equal to

$\boxed{\#\text{ intersections here}}$



$p^{-1}(\beta)$

if β is not primitive one has to multiply the # of intersections by its exponent.