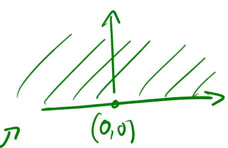
Exercises : week - 1

Choose your favourite exercises and solve those.
Please write what exercises you are attempting to do

Exercise 1
- Let $(\Omega_1, \langle \cdot, \cdot \rangle^{\Omega_1})$ $(\Omega_2, \langle \cdot, \cdot \rangle^{\Omega_2})$ be two domains with Riemannian metrics.
  Show that if $F: \Omega_1 \to \Omega_2$ is bijective and Riemannian-isometric then it is an actual isometry (i.e. $d_1(p,p') = d_2(F(p), F(p'))$ $\forall\, p,p' \in \Omega_1$)

  (the converse is also true: any isometry must be smooth and Riemannian-isometric. You don't have to prove this)

- The above is clearly not true if F is not injective. Show that it can also fail if F is not surjective. (ie. there can be $p,p' \in \Omega_1$ such that $d_2(F(p), F(p')) \neq d_1(p,p')$)



p s.t. $p_y > 0$

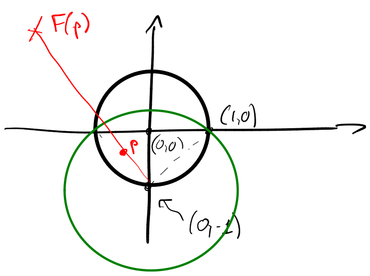
Exercise 2) Let Ω_2 be the upper half-plane and $\langle \cdot,\cdot \rangle_p^{\Omega_2} = \frac{1}{p_y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $p = (p_x, p_y)$

Show that the map
$$D_1 \longrightarrow \Omega_2$$
$$(x,y) \longmapsto \left(\frac{2x}{x^2+(y+1)^2}, \frac{2(y+1)}{x^2+(y+1)^2} - 1 \right)$$
is an isometry when D_1 is equipped with the Poincaré metric

$D_1 = \mathbb{D}$ is the disk of radius 1

making it the Poincaré disk.

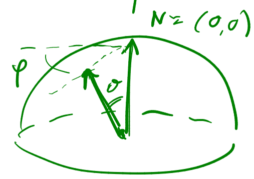
Fun fact this map is obtained via a spherical inversion w.r.t a circle of centre $(0,-1)$



(Note that this map inverts the orientation)

Exercise 3) Write the parametrization of the hemi-sphere in polar coordinates and show that it is isometric to the one I gave you.

(part of the exercise is making sense of its statement)



Exercise 4) Check that the Poincaré disk has infinite diameter and that it is complete

(hence Hopf-Rinow applies)

(*) Characterize which curves are geodesics.

This is probably hard. I don't think you can easily do it with what I said in class.