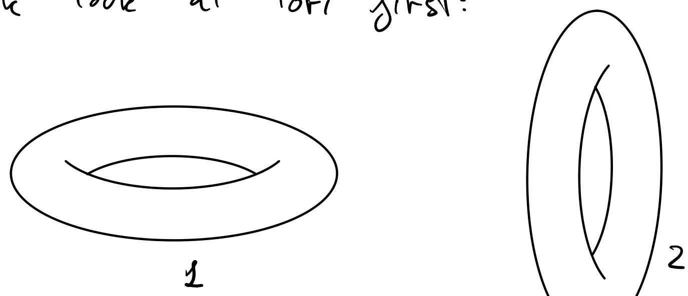
About Teichmüller & Thurston

In this chapter we will only care about surfaces with genus $g \geq 2$ (unless I specifically say otherwise)

§1 Teichmüller space

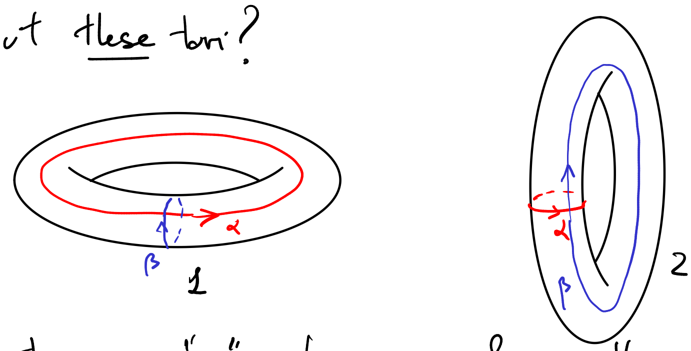
Like the theory of moduli, Teichmüller theory is concerned with understanding the hyperbolic/conformal structures (we know that the two are equivalent by Exercise Sheet 8) that can be put on a surface Σ. To give an idea of the difference we give a quick look at tori first:

Question: are these two tori "the same"?



Well, one wants to say "yes", because they are just the same picture but rotated.

Question 2: how about these tori?



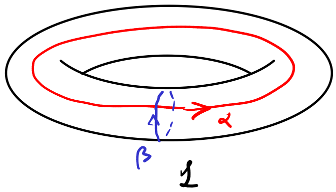
One would now want to say "no" because from the point of view of the tori the marked curves α and β look very different in the two cases.

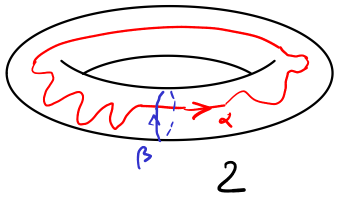
This is the fundamental difference between the space of moduli on Σ and the Teichmüller space of Σ: The former only asks what are the hyperbolic structures on Σ (up to isometry) while the latter investigates marked hyperbolic structures (up to isometries that are "compatible with the marking")

we still have to say what we mean by this.

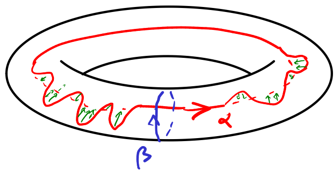
Slogan: the moduli space asks about hyperbolic structures "abstractly" while the Teichmüller spaces looks at hyperbolic structures on a specific, concretely specified, surface.

Question 3: are these equivalent?





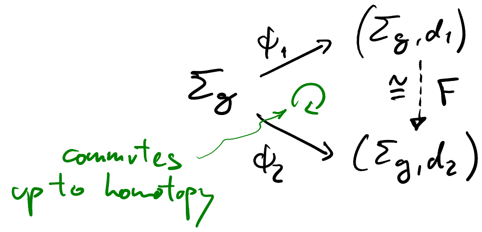
We decide that "yes", because (2) is only a slightly deformed version of (1) and we can recover (1) by "untwisting" the torus a little bit.



More precisely, (1) and (2) look the same "up to homotopy". This is the notion that we want to capture.

Def Fix a topological surface Σ_g. A marked hyperbolic structure is a homeomorphism $\phi: \Sigma_g \longrightarrow (\Sigma_g, d)$ where (Σ_g, d) is a hyperbolic surface.

Two marked hyperbolic structures $\phi_1: \Sigma_g \longrightarrow (\Sigma_g, d_1)$ $\phi_2: \Sigma_g \longrightarrow (\Sigma_g, d_2)$ are equivalent if there exists an isometry $F: (\Sigma_g, d_1) \longrightarrow (\Sigma_g, d_2)$ (i.e. $d_1 = F^* d_2$ as Riemannian metrics) such that ϕ_2 and $F \circ \phi_1$ are homotopic:

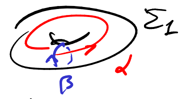


The Teichmüller space of Σ_g is

$$\boxed{\mathrm{Teich}(\Sigma_g) := \{\text{marked hyperbolic structures } \phi: \Sigma_g \longrightarrow (\Sigma_g, d)\} / \text{equivalence}}$$

Rmk 1) This definition of $\mathrm{Teich}(\Sigma_g)$ is only valid for $g \geq 2$ (for $g \leq 1$ there are no hyperbolic structures). The Teichmüller space of the torus should be defined using marked flat structures. If one wants to have a unified definition it is better to say that $\mathrm{Teich}(\Sigma)$ is the space of marked conformal structures up to equivalence.

Rmk 2) This approach is really equivalent to the heuristic discussion.
When we fix Σ_1 we can also fix two curves α and $\beta \subset \Sigma_1$



If we have a marking $\phi: \Sigma_1 \to (\Sigma_1, d)$ we can then look at $\phi(\alpha)$ and $\phi(b)$ (these are the α's and β's that we drew before).
If two markings are equivalent we must have $F: (\Sigma_1, d_1) \to (\Sigma_1, d_2)$ such that $\boxed{F \circ \phi_1(\alpha) \sim \phi_2(\alpha)}$ and $\boxed{F \circ \phi_1(\beta) \sim \phi_2(\beta)}$.
The "rotation" was an isometry between two tori, but it was not compatible with the marking.
(Vice-versa, one can check that if F sends those curves to the correct curves then it does indeed induce an equivalence of markings)

Rmk 3) As a set, the space $\mathrm{Teich}(\Sigma_g)$ does not depend on the choice of the fixed topological surface used to define the markings.
That is, if Σ and Σ' are two fixed surfaces and $G: \Sigma \to \Sigma'$ is a homeo, then we have a bijection

$$\{\phi: \Sigma \to (\Sigma_g, d)\} \xleftrightarrow{1\text{-}1} \{\phi': \Sigma' \to (\Sigma_g, d)\}$$
$$\phi \longmapsto \phi \circ G^{-1}$$
$$\phi' \circ G \longleftarrow\!\shortmid \phi'$$

In oder words, to define the Teichmüller space it is necessary to fix a specific topological surface Σ, but the end result only depends on the homeomorphism class of Σ.

A slightly different point of view is as follows.
When we defined the moduli space of a surface Σ we said that it is "the set of complex structures that can be put on Σ" and we wrote it as $\{\text{Riemann surfaces homeo to } \Sigma\}/\sim$.
This approach is similar to taking marked surfaces.
Instead, we could have considered a quotient of the actual set of complex structures on Σ (i.e. Σ is fixed, and a complex structure is just a set of maps $P(\Sigma) \to \mathbb{C}$).
This is even more clear if we are looking at hyperbolic metrics:

$$\{d: \Sigma \times \Sigma \to \mathbb{R} \mid \text{hyperbolic metric (i.e. st } \tilde{\Sigma} \cong \mathbb{H}^2)\}.$$

this is a respectable set. We don't need to consider markings

This description is highly redundant, because there are going to be many different metrics that produce isometric surfaces (this would correspond to modifying the marking), and we want to classify hyperbolic metrics up to isometry. To do this, we want to identify those metrics that yield isometric results. That is, we say that d_1 and d_2 are equivalent if (Σ, d_1) and (Σ, d_2) are isometric (i.e. $\exists F: \Sigma \to \Sigma$ s.t. $d_1 = F^* d_2$) and define

$$\boxed{\text{Moduli Space}(\Sigma) = \{ d \mid \text{hyp. metric on } \Sigma \} /_{\sim_M}}$$

We can use the same approach for the Teichmuller space as well and define it as some quotient of the set of hyperbolic metrics on Σ. The point now is that we don't want to consider all the isometries, because this would mess up with the marking. Instead, we want to consider isometries that are homotopic to the identity.

I.e $d_1 \sim_T d_2$ if $\exists F: \Sigma \to \Sigma$ homotopic to id_Σ s.t. $d_1 = F^* d_2$.

$$\boxed{\text{Teich}(\Sigma) := \{ d \mid \text{hyp. met. on } \Sigma \} /_{\sim_T}}$$

is an equivalent definition for the Teichmuller space

Take-away: If $F: \Sigma \to \Sigma$ is a homeo that is not homotopic to id_Σ then (Σ, d) and $(\Sigma, F^* d)$ will give different elements of the Teichmüller space (but they will identify the same element in the Moduli space)

Prop/Ex The mapping class group $MCG(\Sigma)$ admits a natural action on $\text{Teich}(\Sigma)$. The quotient $\boxed{\text{Teich}(\Sigma)/_{MCG(\Sigma)}}$ is the Moduli Space of Σ.

This fact is important for at least two reasons. In fact, it turns out that $\text{Teich}(\Sigma)$ is a rather reasonable (topological) space. One can thus try to

- Study Moduli Space (Σ) in terms of the action $MCG(\Sigma) \curvearrowright \text{Teich}(\Sigma)$
- Study $MCG(\Sigma)$ itself " " " " " " "

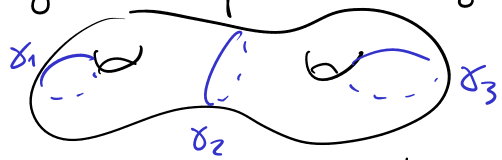
Both these objects are interesting but very complicated. the Teichmuller Space can be used to say non-trivial things about them. (we will later see something on this regard)

§2: Fenchel-Nielsen Coordinates

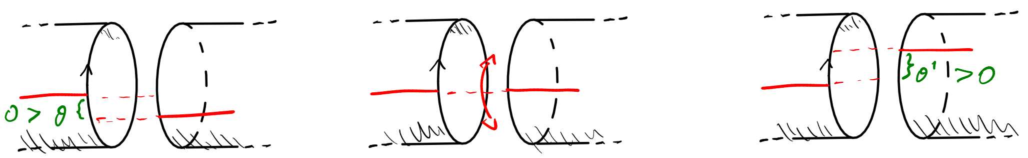
In this section we want to show that there is a natural way to identify $\mathrm{Teich}(\Sigma_g)$ with $\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$.

Idea: say that I want to describe how to construct a specific hyperbolic surface (Σ, d). I can start by choosing a pants decomposition μ and tell you what is the length of the geodesic representative of each curve $\gamma_i \in \mu$

(this is like choosing a point in $\mathbb{R}^{3g-3}_{>0}$)



Since we know that $\exists!$ pair of pants with specified edge-length, this make us think that all that remains to choose to describe completely (Σ, d) is to say "at what angle" you should glue the pants:

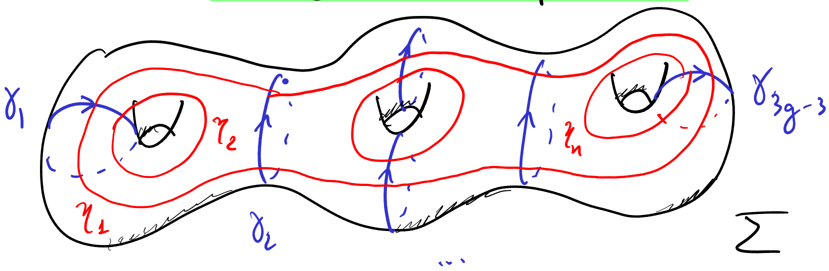


This glueing parameter can be specified by choosing a point in $\mathbb{R}^{3g-3}$.

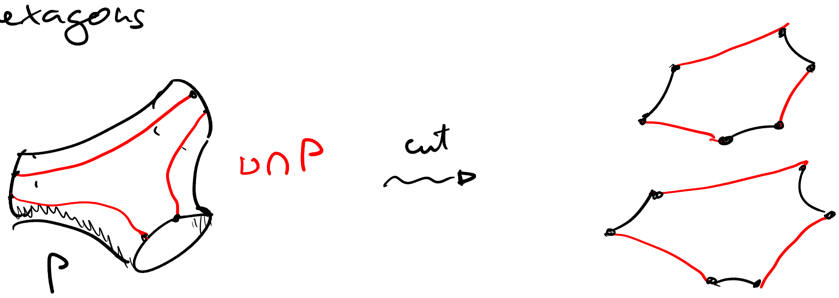
One might expect that changing the glueing angle by $\boxed{2\pi}$ doesn't do anything because the outcomes are clearly isometric. It is in fact true that doing this produces the same element in the moduli space, but this screws up the marking and hence produces different elements in Teich.

We are now going to formalize this idea. It will require some work, because "describing the angle" is kind of tricky. The outcome is pretty neat though...

Set-up: $\boxed{\text{fix}}$ a topological surface $\Sigma = \Sigma_g$ $g \geq 2$, an oriented maximal multicurve $\mu = (\gamma_1, \dots, \gamma_{3g})$ and an auxiliary (unoriented) multicurve $\nu = (\eta_1, \dots, \eta_n)$ that give rise to a $\boxed{\text{hexagon decomposition}}$ of Σ

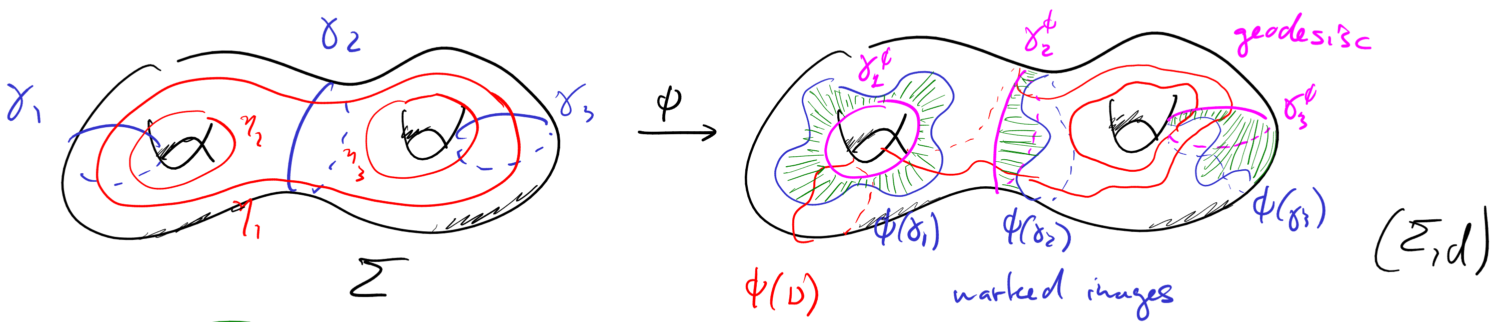


That is, the multicurve ν is chosen in such a way that if we cut Σ along μ and we look at the intersection of ν with any of the pants P in the pants decomposition, we get three curves joining the boundary components of P (so that if we cut Σ along μ and ν we obtain a collection of $6g-6$ hexagons)



(note that every curve in μ is crossed exactly twice by curves in ν)

Let $\phi : \Sigma \to (\Sigma, d)$ be a marked hyperbolic metric on Σ, and consider the multicurves $\phi(\mu)$ and $\phi(\nu)$.
(they still give rise to a hexagon decomposition)
For each $\gamma \in \mu$, $\phi(\gamma)$ has a unique geodesic representative γ^ϕ



Thus canonically identifies $3g-3$ real parameters: $\boxed{(\|\gamma_1^\phi\|, \dots, \|\gamma_{3g-3}^\phi\|) \in \mathbb{R}_{>0}^{3g-3}}$
These are the length parameters associated with μ and the marking $\phi : \Sigma \to (\Sigma, d)$.

we say "canonical" because we fixed μ (the parameters clearly depend on this choice)

It was an exercise to show that the geodesics γ_i^ϕ are in minimal position. In particular we deduce that $\boxed{\mu^\phi = (\gamma_1^\phi, \dots, \gamma_{3g-3}^\phi)}$ is a maximal multicurve and hence gives a pants decomposition. This implies that cutting (Σ, d) along μ^ϕ gives a decomposition into $\boxed{\text{hyperbolic}}$ pants $P_1, \dots, P_g$ with geodesic boundary.

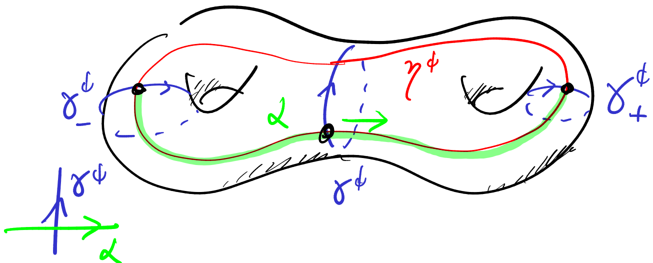
We now want to describe the "angle" at which these hyperbolic pants are attached and thus obtain $3g-3$ torsion parameters. This is where we need the auxiliary multicurve ν.

We can homotope $\phi(\nu)$ to a multicurve $\boxed{\nu^\phi = (\eta_1^\phi, \dots, \eta_n^\phi)}$ such that μ^ϕ and ν^ϕ give a decomposition into hexagons of Σ (e.g. because $\phi(\mu) \sim \mu \Rightarrow$ they are isotopic. Applying the same isotopy to ν produces an adequate ν^ϕ)

(as a matter of fact, we can let η^ϕ be the unique geodesic homotopic to η. Note: μ and ν are in minimal position by the bi-gon criterion, and μ^ϕ, ν^ϕ are in minimal positions because they are geodesic. It follows that (μ, ν) and (μ^ϕ, ν^ϕ) are isotopic (we will need a similar fact lateron). In particular (μ^ϕ, ν^ϕ) would give a hexagon decomposition.)

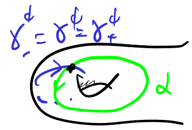
Let $\gamma^\phi \in \mu^\phi$ and choose a subsegment $\boxed{\alpha}$ of a curve $\eta^\phi \in \nu^\phi$, such that α intersects γ^ϕ once, has endpoints in μ^ϕ and does not intersect μ^ϕ anywhere else (note that there are exactly two choices for such an α)

Since γ^ϕ and Σ are oriented, this induces an orientation on α, such that it crosses γ^ϕ from the left to the right.



Let $\boxed{\gamma_-^\phi}$ and $\boxed{\gamma_+^\phi}$ be the curves that α meets before and after γ^ϕ

(note that γ_-^ϕ and γ_+^ϕ may coincide, and they may also coincide with γ^ϕ



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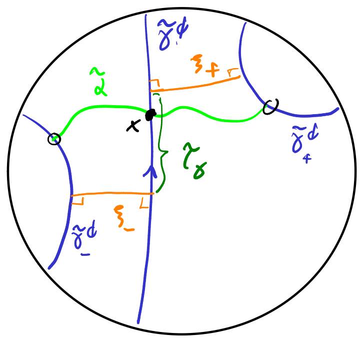
Now, consider the universal cover $p: \mathbb{H}^2 \to \Sigma$ and fix $\boxed{x \in p^{-1}(\gamma^\phi \cap \alpha)}$

Let $\boxed{\tilde\gamma^\phi}$ be the infinite lift of γ^ϕ based at x, and let $\boxed{\tilde\alpha}$ be the lift of α.

Let $\boxed{\tilde\gamma_+^\phi}$ and $\boxed{\tilde\gamma_-^\phi}$ be the infinite lifts of γ_-^ϕ and γ_+^ϕ that start at the endpoints of $\tilde\alpha$.

Finally, let $\boxed{\xi_-}$ and $\boxed{\xi_+}$ be geodesics in $\mathbb{H}^2$ realizing the distance between $\tilde\gamma_-^\phi$ and $\tilde\gamma^\phi$ and γ_+^ϕ and γ^ϕ.



We define the torsion parameter of γ to be the distance between them:

$$\boxed{\tau_\gamma := \pm d_{\mathbb{H}^2}(\xi_-, \xi_+) \in \mathbb{R}}$$

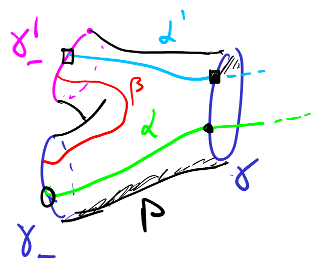
the choice of sign depends on whether ξ_+ comes before or after ξ_- ($\tilde\gamma^\phi$ is oriented)

Claim: τ_γ does not depend on the choices we made.

pf: we chose: the base point x, the subsegment $\alpha \subset \eta^\phi$, a representative η^ϕ homotopic to η.

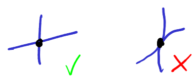
- If x' is a different point in $p^{-1}(\gamma^\phi \cap \alpha)$ then $\exists F \in \mathrm{Aut}(\mathbb{H}^2 \to \Sigma)$ s.t. $F(x) = x'$. It is easy to check that such F sends $\tilde\alpha, \tilde\gamma^\phi, \tilde\gamma^\phi_-, \tilde\gamma^\phi_+$ to the relevant lifts based at x'. Since F is an isometry, the distance $\boxed{\tau_\gamma \text{ remains the same}}$.

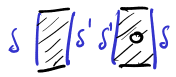
- If α' is the other possible choice, let P be the the pants to the left of γ.
  Let $\beta \subset P$ be the last curve in the hexagon decomposition.
  (i.e s.t $P \cap \nu^\phi = \beta \sqcup \alpha \sqcup \alpha'$)



Claim there is an isotopy of P (which need not fix ∂P pointwise) sending α, α', β to geodesics ξ, ξ', ξ'' realizing the distance between the boundary components of P.

i.e. we can "rotate" around each comp. of ∂P

Sketch of pf: any two simple curves joining the same boundary components of P are homotopic (let δ, δ' be two such curves. Up to isotopy they are smooth and intersect transversely 
If $\delta \cap \delta' \neq \emptyset$ $\exists$ a first intersection, this gives rise to a bi-gon 

by sliding δ' across the bi-gon we can remove this intersection without adding any extra one (here we are "rotating" around ∂P). Iterating, we can assume that δ and δ' are disjoint. P cut along δ and δ' is (a strip) $\sqcup$ (a strip with a hole): 

δ and δ' are homotopic)

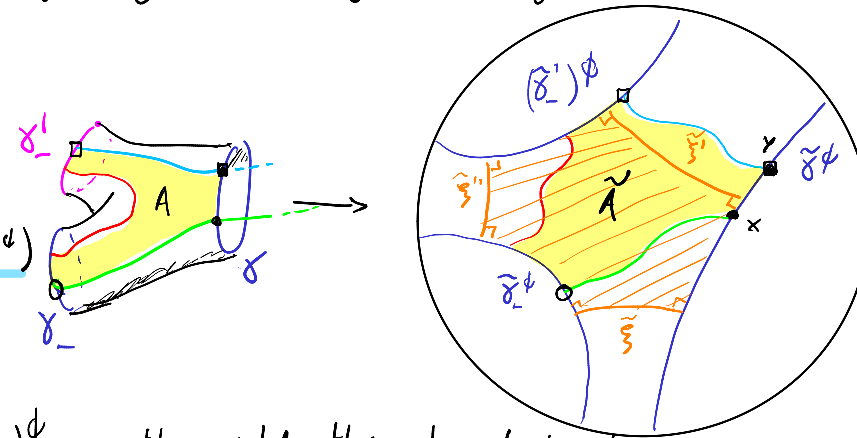
It follows that (α, α', β) is homotopic to (ξ, ξ', ξ'') as a multicurve.

Fact homotopic multicurves are isotopic also when the multicurves contain (non-closed) simple curves with endpoints in $\partial\Sigma$. (Here the isotopy need not fix $\partial\Sigma$ pointwise)

$\Rightarrow$ (α, α', β) is isotopic to (ξ, ξ', ξ'') in P $\square$

Fix now $x \in \tilde{p}^{-1}(\gamma^\phi \cap \alpha)$, let $\boxed{A}$ be a hexagon cut out by $P \setminus (\alpha \cup \alpha' \cup \beta)$ and let $\boxed{\tilde{A} \subseteq \mathbb{H}^2}$ be a lift of that hexagon starting at x.

Note that the lift of α' that we obtain is precisely what we get by choosing an appropriate base point $y \in \tilde{p}^{-1}(\alpha' \cap \gamma^\phi)$ (we can choose whichever base pt we want to define the torsion)

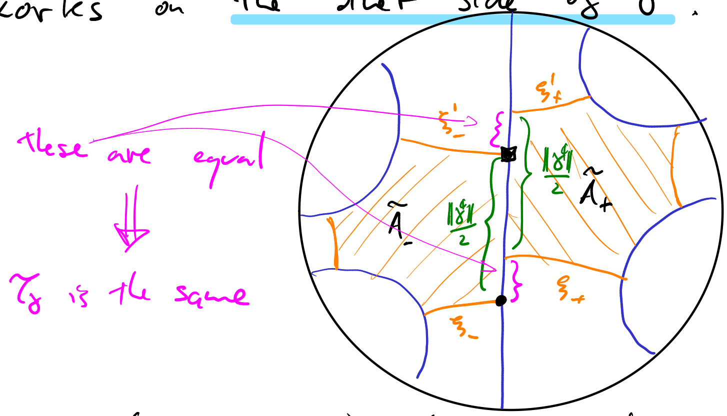


In particular, the curve $(\tilde\gamma'_-)^\phi$ is the right thing to look at, and we are interested in the geodesic $\boxed{\tilde\xi'}$ realizing the distance between $(\tilde\gamma'_-)^\phi$ and $\tilde\gamma^\phi$

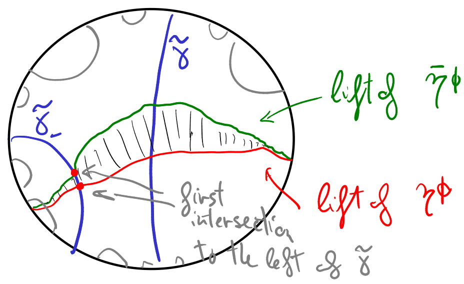
The isotopy sending (α, α', β) to (ξ, ξ', ξ'') lifts to an isotopy sending the hexagon $\tilde{A}$ to a right-angled geodesic hexagon.

This lift fixes the geodesics $\tilde\gamma^\phi$, $(\gamma_-)^\phi$, $(\gamma'_-)^\phi$ (as sets, not pointwise) because it's the lift of an isotopy. It follows that the resulting geodesic hexagon has edges $\tilde\xi, \tilde\xi', \tilde\xi''$ (← these are the curves realizing the distance between the biinfinite geodesics).

By our study of decompositions of pants into right angled hexagons, we deduce that $\boxed{d(\xi, \xi') = \frac{\|\gamma^\phi\|}{2}}$. The same argument works on the other side of γ^ϕ. Thus, the picture we get is:



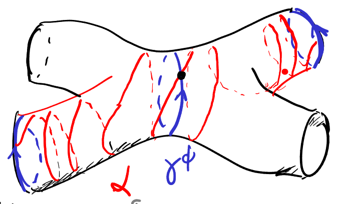
Finally, if $\bar\eta^\phi$ is a different (but homotopic) choice for η^ϕ then the relevant lifts of η^ϕ and $\bar\eta^\phi$ converge to the same points in $\partial\mathbb{H}^2$. Since they are both in minimal position with respect to μ, one can show that the order in which they cross curves in μ must be the same. From this, it follows that they both identify the same curves $\tilde\gamma_-, \tilde\gamma_+$

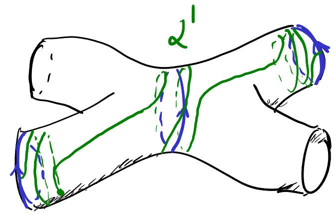


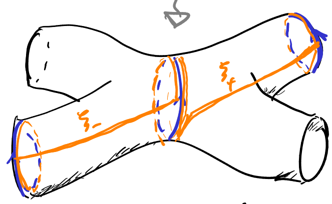
□

Two-ways picture of the torsion parameter:

we look at how much it turns around here





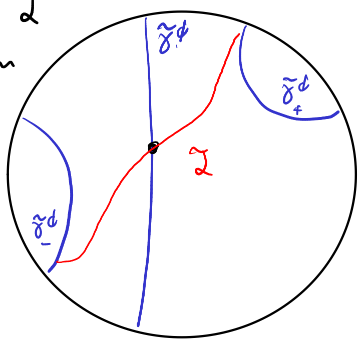


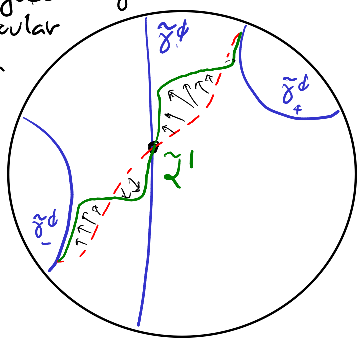
(1) Start off with geodesics γ^ϕ γ_-^ϕ γ_+^ϕ and a curve α crossing them

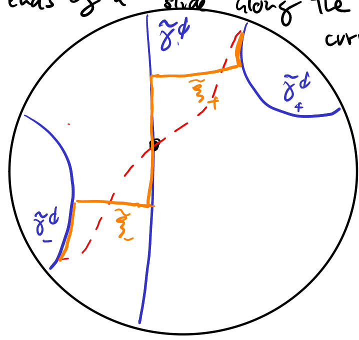
used to be light green, but red is more visible

(2) begin to deform α to make it into a piecewise geodesic that either goes along the γ^ϕ's or is perpendicular to them

(3) in the end we only care about how much does the limit curve walk along γ^ϕ. You can also imagine that we are letting the ends of α' "slide" along the boundary curves γ_-^ϕ γ_+^ϕ







Cool, now we know that every time we have a marking $\phi : \Sigma \to (\Sigma, d)$ we can associate with it $6g-6$ parameters.

Lemma if $\phi_1 : \Sigma \to (\Sigma, d_1)$ and $\phi_2 : \Sigma \to (\Sigma, d_2)$ are equivalent markings, then they have the same parameters.

Pf: We have fixed our multicurves μ and ν.
Say that $F : (\Sigma, d_1) \to (\Sigma, d_2)$ is an isometry making ϕ_1 and ϕ_2 equivalent (i.e. s.t. $\phi_2 \sim F \circ \phi_1$ are homotopy equivalent).

Since F is an isometry, $F(\mu^{\phi_1})$ is composed by geodesics and is homotopic to μ^{ϕ_2}. By uniqueness of geodesics in a given homotopy class, we get that $\boxed{F(\mu^{\phi_1}) = \mu^{\phi_2}}$ and hence the $3g-3$ length parameters of ϕ_1 and ϕ_2 are the same.

Since F is a homeomorphism, $F(\mu^{\phi_1})$ and $F(\nu^{\phi_1})$ still give a hexagon decomposition. As $\boxed{F(\nu^{\phi_1}) \sim \phi_2(\nu)}$, we have that we may as well let $\boxed{\nu^{\phi_2} := F(\nu^{\phi_1})}$. Since F is an isometry, it immediately follows that the torsion parameters coincide as well. $\square$

Since this parameters do not depend on the equivalence class of the marking, they descends to a map of the Teichmüller space

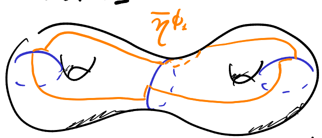
Thm (Fenchel-Nielsen) the length and torsion parameters associated with the multicurves μ, ν on Σ induce a bijection

$$\boxed{\mathrm{Teich}(\Sigma) \xrightarrow{1-1} \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}}$$

proof We showed the map is well defined. Thanks to our explicit construction of hyperbolic metrics, it is very easy to see that it is also surjective.

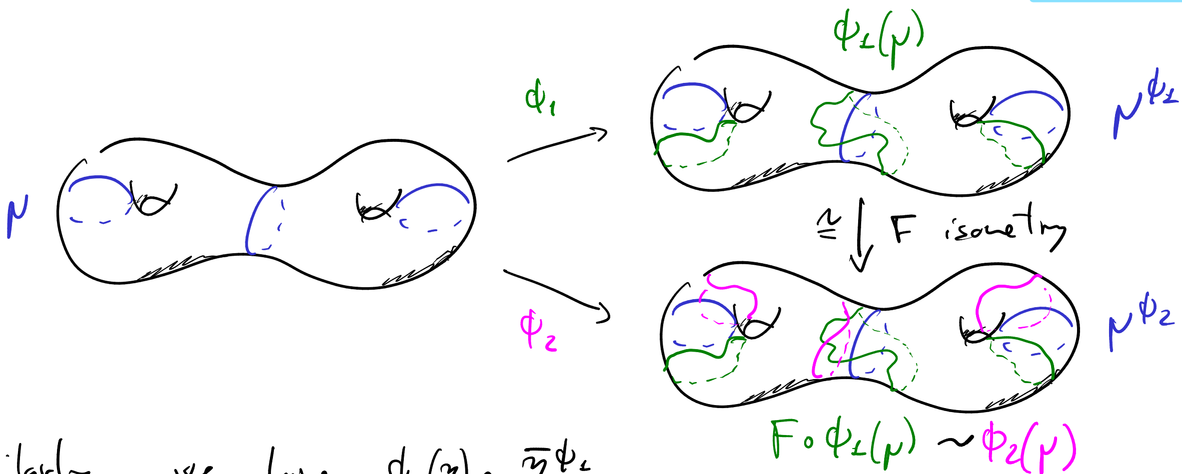
The injectivity is more tricky. Say that $\phi_1 : \Sigma \to (\Sigma, d_1)$ and $\phi_2 : \Sigma \to (\Sigma, d_2)$ have the same parameters. Then the pants in the decomposition given by μ^{ϕ_1} and μ^{ϕ_2} are pairwise isometric because they have the same length parameters.

Let $\boxed{\bar\eta^{\phi_1}}$ and $\boxed{\bar\eta^{\phi_2}}$ be the piecewise-geodesic curves as described in the pictorial description of torsion.



Since the torsion parameters are the same, we can "glue" the isometries between pants in such a way that the curves $\bar\eta^{\phi_1}$ are sent to $\bar\eta^{\phi_2}$. This way we obtain an isometry $F : (\Sigma, d_1) \to (\Sigma, d_2)$

It remains to check that $\phi_2 \sim F \circ \phi_1$. Note that $\boxed{F \circ \phi_1(\mu) \sim \phi_2(\mu)}$:



Similarly, we have $\phi_1(\eta) \sim \bar\eta^{\phi_1}$

$$\Rightarrow \boxed{F \circ \phi_1(\eta) \sim F(\bar\eta^{\phi_1}) = \bar\eta^{\phi_2} \sim \phi_2(\eta)}$$

The proof now follows from the following:

Fact Let $F, F' : \Sigma \to \Sigma$ be homeomorphisms and μ, ν multicurves that decompose Σ into hexagons. If $F(\mu) \sim F'(\mu)$ and $F(\nu) \sim F'(\nu)$, then $\boxed{F \sim F'}$.

Sketch of pf: We know that $F(\mu) \sim F'(\mu) \Rightarrow$ they are isotopic (homotopy $\Rightarrow$ isotopy for multicurve). Let ν' be the image of $F(\nu)$ under this isotopy. It gives an hexagon decomposition with $F'(\mu)$. The proof of homot. $\Rightarrow$ isot. can be improved to show that there is an isotopy $\nu' \sim F'(\nu)$ that fixes $F'(\mu)$ (as a set).

$\Rightarrow$ We get an isotopy that sends $(F(\mu), F(\nu))$ to $(F'(\mu), F'(\nu))$. Hitting F with this isotopy, we get something that coincides with F' on $\mu \cup \nu$. Since what remains is a union of discs (hexagons), we can then homotope F to F' there as well. □

We can use this identification $\mathrm{Teich}(\Sigma) \leftrightarrow \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$ to topologize the Teichmuller space

Fact The topology thus obtained does not depend on the choice of μ and ν

There are indeed many other ways to put a topology on $\mathrm{Teich}(\Sigma)$, and they all coincide. We will a few more of those later.

Cor $\mathrm{Teich}(\Sigma)$ is a contractible topological space (it is homeomorphic to $\mathbb{R}^{6g-6}$)

§3: More topology on Teich

The starting point in the definition of the Fenchel-Nielsen coordinates was the observation that if we fix a multicurve μ in Σ then the length parameters associated with a marking $\phi : \Sigma \to (\Sigma, d)$ are invariant under equivalence of markings.

Let $\mathcal{C}(\Sigma)$ be the family of unoriented closed curves in Σ up to homotopy

$$\boxed{\mathcal{C}(\Sigma) = \{ S^1 \to \Sigma \} /_{\substack{\text{homotopy} \\ \& \text{ inversion}}}}$$

functions $\mathcal{C}(\Sigma) \to \mathbb{R}$

Then we have a natural map

$$\begin{aligned} \mathrm{Teich}(\Sigma) &\xrightarrow{\mathcal{L}en} \mathbb{R}^{\mathcal{C}(\Sigma)} \\ \phi : \Sigma \to (\Sigma, d) &\longmapsto (\gamma \longmapsto \| \gamma^{\phi} \|) \end{aligned}$$

length of the geod. rep. of $\phi(\gamma)$ in (Σ, d)

Thm $\quad \mathcal{L}en : \mathrm{Teich}(\Sigma) \to \mathbb{R}^{\mathcal{C}(\Sigma)}$ is injective

Sketch of pf: it's enough to show that the Fenchel-Nielsen coordinates of $\phi : \Sigma \to (\Sigma, d)$ are uniquely determined by $\mathcal{L}en(\phi) : \mathcal{C} \to \mathbb{R}$. The length parameters are of course determined by $\mathcal{L}en(\phi)$, to show that the torsion parameters are uniquely determined as well we need the following:

Fact $\quad$ Let $\mu = (\gamma_1, \dots, \gamma_{3g-3})$ a multicurve used to define the F-N coordinates and let α_i be a curve such that

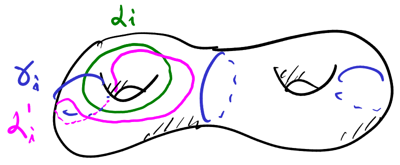
- $i([\alpha_i], [\gamma_i]) > 0$
- $i([\alpha_i], [\gamma_j]) = 0 \quad \forall j \neq i$

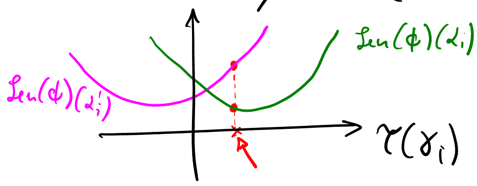
$i([\alpha], [\beta])$ is the minimal number of intersections among curves homotopic to α and β (see the Exercises W.9-10)

Fix the length coordinates and vary the torsion coordinates τ.

Then
- $\mathcal{L}(\alpha_i)$ does not depend on $\tau(\gamma_j) \quad \forall j \neq i$
- $\mathcal{L}(\alpha_i)$ is a strictly convex function of $\tau(\gamma_i)$

The proof of the theorem follows easily, because $\forall \gamma_i \in p$ we can find non homotopic curves α_i, α_i' as in the Fact. It follows from strict convexity that $\tau(\gamma_i)$ is determined by $\mathcal{L}en(\phi)(\alpha_i)$ and $\mathcal{L}en(\phi)(\alpha_i')$





$\square$

<u>Rmk</u> it follows from the pf of the theorem that it is possible to choose $9g-9$ curves such that $\boxed{\mathcal{L}en : \text{Teich} \longrightarrow \mathbb{R}^{9g-9}}$ is still injective.

Note that $\mathbb{R}^{\mathcal{C}}$ is a topological space with the product topology

Th The map $\text{Teich}(\Sigma) \xrightarrow{\mathcal{L}en} \mathbb{R}^{\mathcal{C}(S)}$ is an embedding (i.e. $\text{Teich}(\Sigma)$ is homeomorphic to its image)

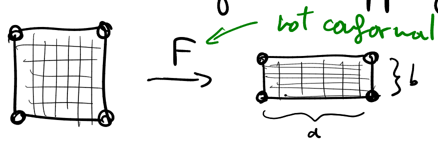
(this theorem is not very hard to prove: since $\mathbb{R}^{\mathcal{C}}$ is second countable, Hausdorff and we already know that $\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3} \overset{\cong}{\longleftrightarrow} \text{Teich} \xrightarrow{\mathcal{L}en} \mathbb{R}^{\mathcal{C}}$ is injective, it is enough to show that it is also continuous and proper.)

Cor We can use the embedding $\mathcal{L}en : \text{Teich}(\Sigma) \longrightarrow \mathbb{R}^{\mathcal{C}(\Sigma)}$ to define a canonical topology on $\text{Teich}(\Sigma)$. This topology coincides with the topology obtained via Fenchel-Nielsen coordinates.

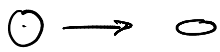
Cor The topology induced by F-N does not depend on the choice of hexagon decomposition.

One more way to topologize $\text{Teich}(\Sigma)$ is by means of the Teichmüller metric.

Intuition: we know that there is no conformal map sending the square to a rectangle mapping corners to corners



Yet the natural map F "stretching" the square to the rectangle is not too bad, in fact it is quasi-conformal.
That is, it sends circles to ellipses that are not too eccentric.



Informal definition F is K-quasi conformal for some $K \geq 1$ if it maps every circle to ellipses of eccentricity at most K
(the eccentricity of an ellipse is the ratio $\boxed{\dfrac{\text{largest radius}}{\text{smallest radius}}}$)

Rmk conformal maps are 1-quasi-conformal.

In our example, F is $\frac{a}{b}$-quasi-conformal. In particular,
the quasi-conformality constant of F describes "how distant is the rectangle from being a square"

Similarly, if $\phi_1 : \Sigma \to (\Sigma, d_1)$ and $\phi_2 : \Sigma \to (\Sigma, d_2)$ are not equivalent markings then we can describe how far they are by saying what is the conformality constant of a homeo $F : \Sigma \to \Sigma$ s.t. $F \sim \phi_2 \circ \phi_1^{-1}$

Def the Teichmüller distance in $\text{Teich}(\Sigma)$ is

$$\boxed{d_{Teich}(\phi_1, \phi_2) := \frac{1}{2} \log \left(\inf \left\{ K \geq 1 \;\middle|\; \begin{array}{l} \exists F : \Sigma \to \Sigma \ \ K\text{-quasi.conf} \\ F \sim \phi_2 \circ \phi_1 \end{array} \right\} \right)}$$

It is not very hard to show that d_{Teich} is a metric. Proving the following is harder:

Thm (Teichmüller) The inf in the def. of d_{Teich} is a minimum and the function realizing it is unique.
Furthermore, $\boxed{(\text{Teich}, d_{Teich})}$ is a complete geodesic metric space.

Fact the Teichmüller metric induces the usual topology on $\text{Teich}(\Sigma)$

§4: Thurston compactification

We now know that there is an embedding $\mathrm{Teich}(\Sigma) \xhookrightarrow{\mathrm{Len}} \mathbb{R}^{\mathcal{C}(\Sigma)}$.
Consider the projectivization $\boxed{\mathbb{P}\mathbb{R}^{\mathcal{C}(\Sigma)} := \left(\mathbb{R}^{\mathcal{C}(\Sigma)} \setminus \{0\}\right)/\mathbb{R}^*}$, and write that of $\mathrm{Len}(\mathrm{Teich}(\Sigma))$

Fact Let Σ be a closed surface of genus $g \geq 2$.
The composition $\boxed{\mathrm{Teich}(\Sigma) \xhookrightarrow{\mathrm{Len}} \mathbb{R}^{\mathcal{C}(\Sigma)} \longrightarrow \mathbb{P}\mathbb{R}^{\mathcal{C}(\Sigma)}}$ is an embedding.
The closure $\boxed{\overline{\mathrm{Teich}(\Sigma)}^{\mathbb{P}\mathbb{R}^{\mathcal{C}(\Sigma)}}}$ is homeomorphic to the closed disc $\boxed{D^{6g-6}}$ (in particular it is compact) and $\mathrm{Teich}(\Sigma)$ is the interior of this disc (i.e. $D^{6g-6} \setminus \mathrm{Teich}(\Sigma) = \partial D^{6g-6} \cong S^{6g-7}$)

Def The space $\overline{\mathrm{Teich}(\Sigma)}^{\mathbb{P}\mathbb{R}^{\mathcal{C}(\Sigma)}}$ is the Thurston compactification of the Teichmüller space.

Rmk It is easy to check that $\mathrm{Teich}(\Sigma) \longrightarrow \mathbb{P}\mathbb{R}^{\mathcal{C}(\Sigma)}$ is injective.
It is more complicated to show that $\overline{\mathrm{Teich}(\Sigma)}^{\mathbb{P}\mathbb{R}^{\mathcal{C}}}$ is compact and it is even harder to show that $\overline{\mathrm{Teich}(\Sigma)} \cong D^{6g-6}$.

We already remarked that $MCG(\Sigma)$ acts on $\mathrm{Teich}(\Sigma)$. More precisely, given $F : \Sigma \circlearrowleft$ homeo we let $\boxed{[F].[\Sigma \xrightarrow{\phi} (\Sigma, d)] := [\phi \circ F^{-1} : \Sigma \longrightarrow (\Sigma, d)]}$

$$\begin{tikzcd} & (\Sigma, d) \\ \Sigma \arrow[ur, "\phi"] \arrow[dr, "F"'] & \\ & \Sigma \arrow[uu, dashed, "\phi \circ F^{-1}"'] \end{tikzcd}$$

we want to compose F to the right because it's a map among topological spaces (before we fix the metric). Then we need to use F^{-1} instead of F in order to get a left action.

Similarly, $MCG(\Sigma)$ acts on $\mathcal{C}(\Sigma)$ and hence on $\mathbb{R}^{\mathcal{C}(\Sigma)}$ by letting
$$\boxed{([F]\cdot f)[\gamma] := f([F^{-1} \circ \gamma]) \qquad \forall f : \mathcal{C}(\Sigma) \longrightarrow \mathbb{R}}$$
This action is by continuous maps and it clearly coincides with the action on $\mathrm{Teich}(\Sigma) \subset \mathbb{R}^{\mathcal{C}(\Sigma)}$. Note also that it descends to an action $MCG(\Sigma) \curvearrowright \mathbb{P}\mathbb{R}^{\mathcal{C}(\Sigma)}$.

Bottom line: The action $MCG(\Sigma) \curvearrowright \mathbb{P}\mathbb{R}^{\mathcal{C}(\Sigma)}$ restricts to an action on the Thurston compactification $MCG(\Sigma) \curvearrowright \overline{\mathrm{Teich}(\Sigma)}^{\mathbb{P}\mathbb{R}^{\mathcal{C}}}$

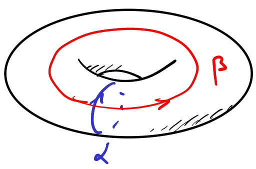
It will be very interesting to study the action $MCG(\Sigma) \curvearrowright \overline{\mathrm{Teich}(\Sigma)}^{\mathbb{PR}^{\mathcal{C}}}$.
In particular, given $[F] \in MCG(\Sigma)$ it will be interesting to study its fixed points set $\boxed{\mathrm{Fix}([F]) \subseteq \overline{\mathrm{Teich}(\Sigma)}^{\mathbb{PR}^{\mathcal{C}}}}$ (note that $\mathrm{Fix}([F])$ cannot be empty because of Brouwer fixed point theorem)
The dynamics of the action $\mathbb{Z} \curvearrowright \overline{\mathrm{Teich}(\Sigma)}^{\mathbb{PR}^{\mathcal{C}}}$ is also very interesting.
$n \mapsto [F]^n \curvearrowright \mathrm{Teich}$

§5: One guiding example

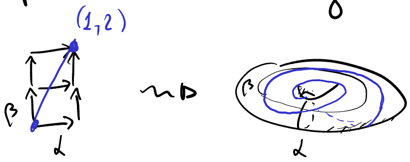
Consider the torus $\mathbb{T} \simeq S^1 \times S^1$ and let

$$\boxed{\mathrm{Teich}(\mathbb{T}) := \{\phi : \mathbb{T} \to (\mathbb{T}, d) \mid \underbrace{d \text{ Euclidean metric}}_{\text{i.e. } \tilde{\mathbb{T}} \cong \mathbb{R}^2} \text{ s.t. } \underbrace{\mathrm{Area}(\mathbb{T}, d) = 1}_{\text{we didn't need this condition for } \mathrm{Teich}(\Sigma) \text{ because of Gauss-Bonnet}} \}/\sim}$$

(this is the correct analogue of the Teichmüller space for flat metrics)

After we choose two curves like so: 

.
we can identify $\mathcal{C}(\mathbb{T})$ with $\mathbb{Z}^2$
(As a set, $\mathcal{C}(\Sigma)$ is equal to $\pi_1(\Sigma)/_{\text{conjugation}}$. $\pi_1(\mathbb{T}) \cong \mathbb{Z}^2 \overset{\text{Abelian}}{\Longrightarrow} \mathcal{C}(\mathbb{T}) \cong \mathbb{Z}^2$.
Picking α, β corresponds to choosing a basis of $\mathbb{Z}^2$)



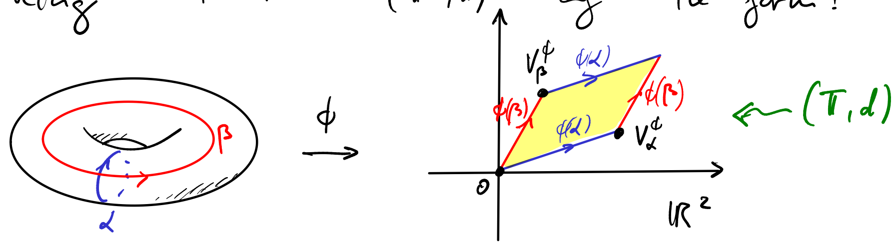
Consider the matrix $\boxed{A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})}$

Let $F : \mathbb{T} \to \mathbb{T}$ be a homeo such that $\boxed{[F^{-1}] \cdot \begin{pmatrix} a \\ b \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}}$ $\forall a, b \in \mathbb{Z}$
here we are identifying $\mathcal{C}(\mathbb{T})$ with $\mathbb{Z}^2$

(e.g. let $F = T_\beta^{-2} \circ T_\alpha^{-1}$: $T_\alpha \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2\mathbb{Z}$, $T_\beta^2 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2\mathbb{Z}$ & $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ $\Longrightarrow$ That F works)

Rmk I could have used $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ as well, but then some special numbers appear and I decided it was good to use something more "generic"

Consider a marking $\phi: \mathbb{T} \longrightarrow (\mathbb{T}, d)$ of the form:



V_α^ϕ and V_β^ϕ are some vectors in $\mathbb{R}^2$

That is, $(\mathbb{T}, d)$ is the torus obtained quotienting $\mathbb{R}^2$ by $(z \mapsto z + V_\alpha^\phi)$ and $(z \mapsto z + V_\beta^\phi)$, and ϕ sends α and β to the projections of the segments $\overline{OV_\alpha^\phi}$ and $\overline{OV_\beta^\phi}$.
In other words, the lift $\tilde{\phi}$ of ϕ to $\mathbb{R}^2$ sends α to the segment $\overline{OV_\alpha^\phi}$ and β to $\overline{OV_\beta^\phi}$.
Note that $V_\alpha^\phi, V_\beta^\phi \in \mathbb{R}^2$ must be such that $\boxed{\det(V_\alpha^\phi | V_\beta^\phi) = 1}$ because we are requiring that $(\mathbb{T}, d)$ has area 1

2×2 matrix having V_α^ϕ and V_β^ϕ as columns

Rmk/Ex every marking $\mathbb{T} \to (\mathbb{T}, d)$ is equivalent to a marking as above for an appropriate choice of V_α^ϕ, V_β^ϕ

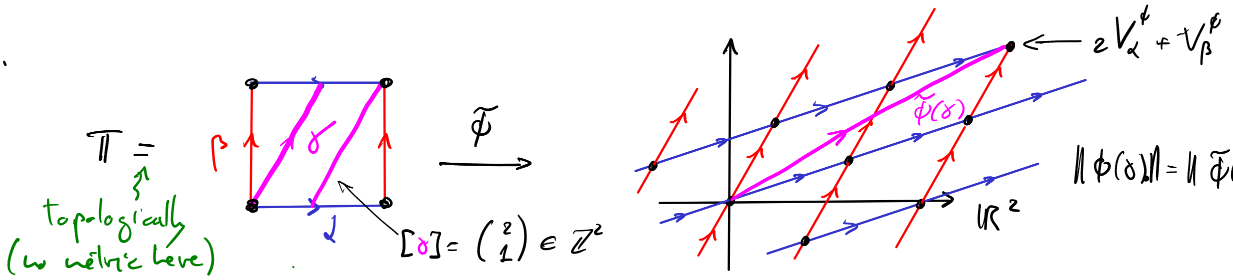
this is short for $\mathcal{L}en([\phi: \mathbb{T} \to (\mathbb{T}, d)])$

We can write explicitly $\mathcal{L}en(\phi): \mathcal{C}(\mathbb{Z}) \longrightarrow \mathbb{R}$:
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \rotatebox{90}{=} \; \mathbb{Z}^2$

$$\boxed{\mathcal{L}en(\phi)\begin{pmatrix} a \\ b \end{pmatrix} = \| a V_\alpha^\phi + b V_\beta^\phi \| = \| (V_\alpha^\phi | V_\beta^\phi) \begin{pmatrix} a \\ b \end{pmatrix} \|}$$

matrix multiplication

E.g.



$\| \phi(\gamma) \| = \| \tilde{\Phi}(\gamma) \| = \| 2V_\alpha^\phi + V_\beta^\phi \|$

Let us now look at $[F^n] \cdot [\phi] = [\phi \circ F^{-n}]$. By our definition of F we have that $F^n: \mathcal{C}(\mathbb{T}) \longrightarrow \mathcal{C}(\mathbb{T})$ sends $\begin{pmatrix} a \\ b \end{pmatrix}$ to $A^n \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}^n \begin{pmatrix} a \\ b \end{pmatrix}$.

Therefore, $$\boxed{\mathcal{L}en([F^n] \cdot [\phi]) \begin{pmatrix} a \\ b \end{pmatrix} = \| (V_\alpha^\phi | V_b^\phi) A^n \begin{pmatrix} a \\ b \end{pmatrix} \| \in \mathbb{R}_{>0}}$$
$$\rotatebox{90}{=} \; ([F^n] \cdot \mathcal{L}en(\phi)) \begin{pmatrix} a \\ b \end{pmatrix}$$

Note that A has eigenvalues $\boxed{\lambda_1 = 2 + \sqrt{3}} > 1$ and $\boxed{\lambda_2 = 2 - \sqrt{3} = \frac{1}{\lambda_1}} < 1$
Fix W_1, W_2 eigenvectors then $\begin{pmatrix} a \\ b \end{pmatrix} = x_1 W_1 + x_2 W_2$ for some $x_1, x_2 \in \mathbb{R} \setminus \{0\}$

(e.g. $W_1 = \begin{pmatrix} 1 \\ \sqrt{3} - 1 \end{pmatrix}$ $W_2 = \begin{pmatrix} \frac{1 - \sqrt{3}}{2} \\ 1 \end{pmatrix}$)

Since the eigenvectors have irrational slope $\begin{pmatrix} a \\ b \end{pmatrix}$ can't be a multiple of either of them (unless $a = b = 0$)

We can easily compute the length of the curve $\binom{a}{b}$ w.r.t the marking $[F^n]\cdot[\phi]$:

$$\mathcal{L}en([F^n]\cdot[\phi])\tbinom{a}{b} \stackrel{!}{=} \| (V_\alpha^\phi | V_\beta^\phi)\, A^n (x_1 W_1 + x_2 W_2)\|$$

$$\stackrel{!}{=} \|(V_\alpha^\phi|V_\beta^\phi)(\lambda_1^n x_1 W_1 + \lambda_1^{-n} x_2 W_2)\|$$

$$\stackrel{!}{=} \underbrace{|x_1|\lambda_1^n \|(V_\alpha^\phi|V_\beta^\phi)W_1\|}_{\substack{\downarrow\, n\to+\infty \\ +\infty}} + \underbrace{|x_2|\lambda_1^{-n}\|(V_\alpha^\phi|V_\beta^\phi)W_2\|}_{\substack{\downarrow\, n\to+\infty \\ 0}} - \underbrace{\sqrt{2|x_1 x_2|\langle (V_\alpha^\phi|V_\beta^\phi)W_1, (V_\alpha^\phi|V_\beta^\phi)W_2\rangle}}_{\text{constant}}$$

If we are interested to $[\mathcal{L}en([F^n][\phi])] \in \mathbb{P}\mathbb{R}^{\mathcal{C}(\pi)}$, we can rescale it by constant. In particular, we can rescale by $\lambda_1^n \|(V_\alpha^\phi|V_\beta^\phi)W_1\|$ and obtain a function in the same projective class $[\mathcal{L}en([F^n][\phi])]$ (it is important that we rescale by a constant that does not depend on $\binom{a}{b}$ (and hence x_1, x_2)). We get:

$$\boxed{\mathcal{L}en([F^n][\phi]) \sim \left[\tbinom{a}{b}\mapsto |x_1| + O\!\left(\tfrac{1}{\lambda_1^n}\right)\right]}$$

(a term that goes to zero as $\frac{1}{\lambda_1^n}$)

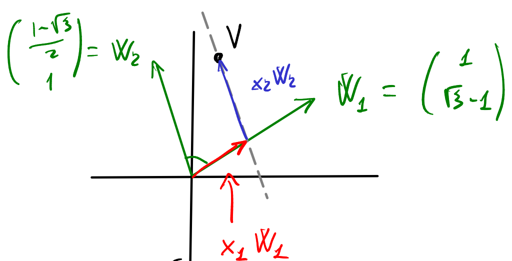
(pointwise convergence)

That is, the sequence $([F]^n\cdot \mathcal{L}en([\phi]))_{n\in\mathbb{N}} \in \mathbb{P}\mathbb{R}^{\mathcal{C}(\pi)}$ converges to the projective equivalence class of the function $\binom{a}{b}\mapsto x_1$.

On the contrary $([F]^{-n}\cdot\mathcal{L}en([\phi]))_{n\in\mathbb{N}}$ converges to $\left[\binom{a}{b}\mapsto |x_2|\right]$

Note that the limit does not depend on the marking ϕ we had chosen! That is, we see that the action $\boxed{\begin{matrix}\mathbb{Z} \curvearrowright \overline{Teich(\pi)}^{\mathbb{P}\mathbb{R}^{\mathcal{C}}} \\ n\mapsto [F]^n \curvearrowright \overline{Teich}^{\mathbb{P}\mathbb{R}^{\mathcal{C}}}\end{matrix}}$ has north-south dynamics.

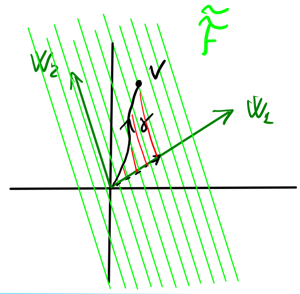
Now, it remains to describe the function $\binom{a}{b}\to x_1$. This is trivial linear algebra, but it is constructive seeing it geometrically:



For every $V\in\mathbb{R}^2$, $x_1(V)$ is the distance of V from the line spanned by W_2 renormalized by some constant depending on $\|W_2\|$ and the angle between W_1 and W_2

(since we care about $[x_1]\in\mathbb{P}\mathbb{R}^{\mathcal{C}(\pi)}$ we can ignore these constants)

In other words, the level sets of the function $x_1 : \mathbb{R}^2 \to \mathbb{R}$ are a $\boxed{\text{foliation } \tilde{\mathcal{F}}}$ of $\mathbb{R}^2$ by lines parallel to W_2.



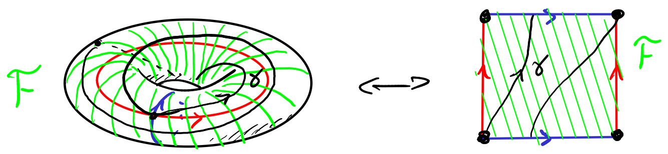
In particular, we can express $x_1(V)$ as the integral over a curve γ going from 0 to V of the progress it made perpendicularly to $\tilde{\mathcal{F}}$ (think of it as "counting" the leaves of $\tilde{\mathcal{F}}$ that we had to cross to get to $\tilde{V}$)

(Formally, $x_1(V) = \int_0^1 \langle \dot{\gamma}(t), \hat{W}_2^{\perp} \rangle \, dt$, where $\gamma : [0,1] \to \mathbb{R}^2$ is any smooth curve with $\gamma(0) = 0$, $\gamma(1) = V$)

unit vector perpendicular to W_2

Why do we do this? Because this last description can be used directly on the torus without going to the universal cover!

That is, $\boxed{\mathcal{F} := p(\tilde{\mathcal{F}})}$ is a foliation of the torus (each leaf is dense because it has irrational slope) and for any curve $\gamma : S^1 \to \mathbb{T}$ $x_1(\gamma)$ is the integral of the progress of γ perpendicular to $\mathcal{F}$.



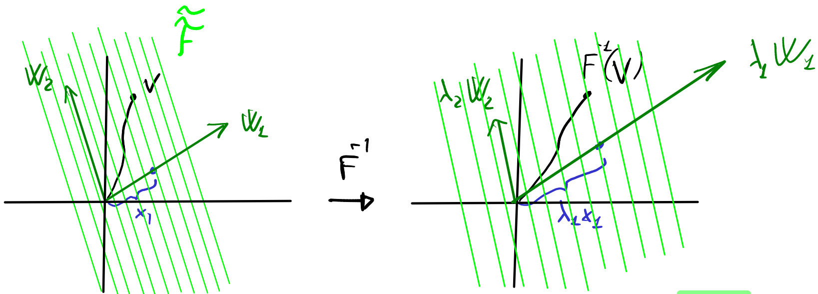
as a function, μ must satisfy a few extra assumptions. In particular, it must be additive and it must be invariant under homotopies that preserve the leaves of $\mathcal{F}$

In other words, we can think of $\mathcal{F}$ as a measured foliation of $\mathbb{T}$.
That is, $\mathcal{F}$ is a foliation equipped with a "transverse measure" i.e. a function $\boxed{\mu : \{\text{curves}\} \to \mathbb{R}}$ such that $\mu(\gamma : [0,1] \to \mathbb{T})$ is the total progress orthogonal to $\mathbb{T}$.

We showed that the function $\boxed{x_1}$ coincides with the measure $\boxed{\mu}$ up to rescaling by a constant.

Finally, note that $\forall \gamma \in \mathcal{C}(\mathbb{T})$, $\boxed{[F] \cdot x_1(\gamma) = x_1([F^{-1} \circ \gamma]) = \lambda_1 \, x_1([\gamma])}$.
(more generally $F_* \mu(\gamma) := \mu(F^{-1} \circ \gamma) = \lambda_1 \, \mu(\gamma) \quad \forall \gamma : [0,1] \to \mathbb{T}$)

In other words, the $\boxed{\text{projective}}$ class of x_1 is fixed by $[F]$, while its actual values are rescaled by λ. This is because F^{-1} is "stretching" the direction W_1, thus "increasing the distance" among leaves of $\mathcal{F}$



$\mathcal{F}$ is called unstable foliation and is denoted by $\boxed{\mathcal{F}_u}$

Unfortunately it appears that there is a bit of convention clash between stable/unstable in the context of homeomorphisms v.s. flows. This is because we had to use F^{-1} instead of F in order to get the desired result.

The same reasoning shows that x_2 is obtained as a measure transverse to the foliation $\boxed{\mathcal{F}_s}$ whose leaves are parallel to W_1.
This foliation is called stable.

Bottom line: The homeomorphism $F: \mathbb{T} \to \mathbb{T}$ comes with two measured foliations of $\mathbb{T}$ $\boxed{(\mathcal{F}_s, \mu_s)}$ and $\boxed{(\mathcal{F}_u, \mu_u)}$
such that:

- $\mathcal{F}_s$ and $\mathcal{F}_u$ are $\boxed{\text{transverse}}$
- $F(\mathcal{F}_u) = \mathcal{F}_u$ $\quad \boxed{F_*(\mu_u) = \lambda_1 \mu_u}$
- $F(\mathcal{F}_s) = \mathcal{F}_s$ $\quad \boxed{F_*(\mu_s) = \lambda_2 \mu_s = \frac{1}{\lambda_1} \mu_s}$

and we showed that $\forall \phi: \mathbb{T} \to (\mathbb{T}, d)$ we have

- $[F^n] \cdot \mathcal{L}en(\phi) \xrightarrow{n \to \infty} [\mu_u]$
- $[F^{-n}] \cdot \mathcal{L}en(\phi) \xrightarrow{n \to \infty} [\mu_s]$

projective classes of functions $C(\mathcal{T}) \to \mathbb{R}$

Def Such a homeomorphism is called Anosov.

§6: Classification of homeomorphisms

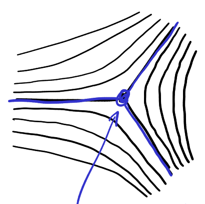
We are interested in describing $\partial \mathrm{Teich}(\Sigma) = \mathbb{S}^{6g-7} \subset \mathbb{D}^{6g-6} \subset \mathbb{P}\mathbb{R}^{C(\Sigma)}$

In the torus case we found elements of $\mathrm{Teich}(\mathbb{T})$ converging to measures transverse to foliations. We would like to see if the same is true on other surfaces.

<u>Problem</u>: a closed surface Σ admits a foliation by lines iff $\boxed{\chi(\Sigma) = 0}$

if $\partial\Sigma = \emptyset$ then $\Sigma = \mathbb{T}$

Solution: we consider singular foliations instead. That is, we allow for pronged singularities:



the possible numbers and types of singularities is determined by $\chi(\Sigma)$

Every surface admits singular foliations. We can also consider foliations that are equipped with a transverse measure, and it is possible to obtain measured foliations as limits of sequences in $\mathrm{Teich}(\Sigma)$. As a matter of fact, the following is true:

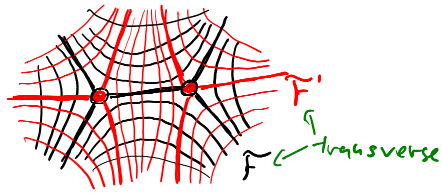
Th (Thurston) in the compactification $\overline{\mathrm{Teich}(\Sigma)}^{\mathbb{R}\mathbb{P}^C} \cong \mathbb{D}^{6g-6}$ one can identify $\partial\mathbb{D}^{6g-6} \cong \mathbb{S}^{6g-7}$ with the space of projective measured foliations:

$$\boxed{\mathcal{PMF} := \left\{ (\mathcal{F}, \mu) \;\middle|\; \begin{array}{l}\mathcal{F} \text{ singular foliation} \\ \mu \text{ transverse measure}\end{array} \right\} \Big/ \sim}$$

classically, transverse measures are defined differently from the way I did it. They are always positive and they are only defined on curves transverse to the foliation. The idea is the same, though...

eq. up to:
* constants
* isotopies
* Whitehead moves

"transverse" means that the foliations have the same set of singularities and are generically transverse everywhere else:



Def a homeomorphism $F:\Sigma\to\Sigma$ is pseudo-Anosov if it is homotopic to a homeomorphism $\bar{F}:\Sigma\to\Sigma$ for which there exist $\lambda>1$ and two transverse measured foliations $(\mathcal{F}_u,\mu_u)$, $(\mathcal{F}_s,\mu_s)$ such that:

- $\bar{F}(\mathcal{F}_u)=\mathcal{F}_u$ $\qquad \bar{F}_*(\mu_u)=\lambda\mu_u$
- $\bar{F}(\mathcal{F}_s)=\mathcal{F}_s$ $\qquad \bar{F}_*(\mu_s)=\frac{1}{\lambda}\mu_s$

This is a fundamental result:

Thm (Nielsen-Thurston Classification) let Σ be a surface of genus $g\geq 2$ and $[F]\in MCG(\Sigma)$. Then one of the following must hold.

- $[F]^n=[id]$ for some $n\in\mathbb{N}$ (periodic)
- $[F]$ fixes a multicurve μ up to homotopy (reducible)
- F is pseudo-Anosov

Further, pseudo-Anosov homeomorphisms are not parabolic nor reducible

Thurston's original proof of this Theorem was obtained by studying the fixed points of $[F]:\overline{\mathrm{Teich}(\Sigma)}^{\mathbb{PR}^{e}}\circlearrowleft$.
Pseudo-Anosov elements are homeomorphisms that give rise to a north-south dynamics. I.e. one can prove that

$$[F^n]\,\mathrm{Len}(\phi)\longrightarrow[\mu_u] \qquad \forall\phi:\Sigma\to(\Sigma,d)$$
$$[F^{-n}]\,\mathrm{Len}(\phi)\longrightarrow[\mu_s]$$

A mind-blowing reason to care about pseudo-Anosov:

Thm (Thurston) let Σ have genus ≥ 2 and fix $F:\Sigma\to\Sigma$.
Then the mapping torus $M_F:=(\Sigma\times[0,1])/(x,1)\sim(F(x),0)$ admits a hyperbolic structure iff F is pseudo-Anosov.