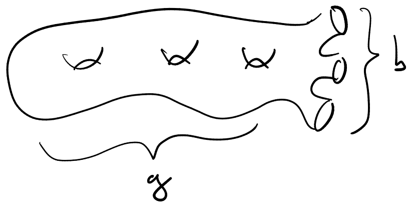
Chapter 4: Topology and Geometry of surfaces

§1: Simple closed curve

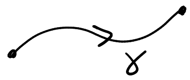
(due to time constraint we will be light on details...)

We will now focus on compact surfaces, possibly with boundary:

$\Sigma_{g,b} =$ 

Recall that a curve on Σ is the image of a continuous parametrized curve $\gamma : I \to \Sigma$

Often our curves will be oriented (i.e. we remember the direction it goes)

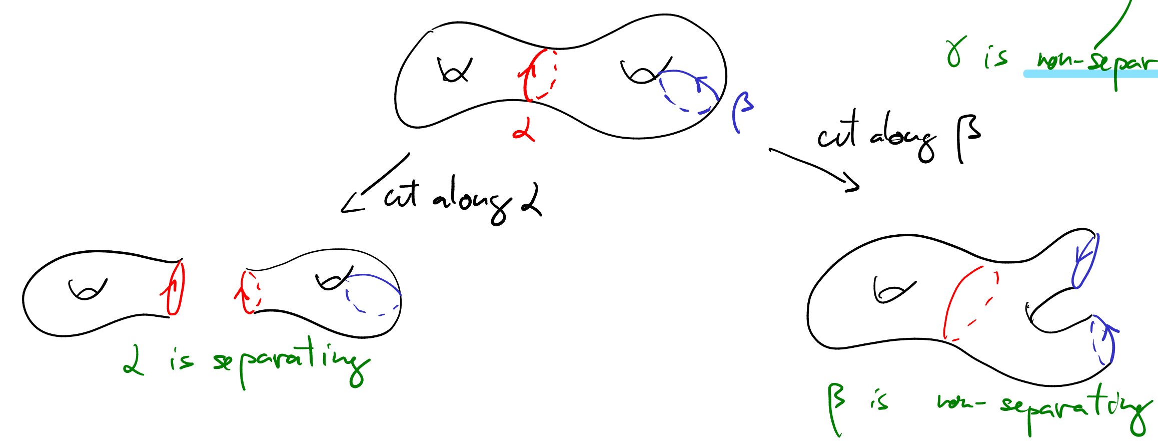


a closed curve is the image of a cts. map $\gamma : S^1 \to \Sigma$

Def a (closed) curve is simple if it comes from an injective parametrized curve.
We will be particularly interested in simple closed curves (s.c.c.)

An important feature of simple closed curves is that if $\gamma \subseteq \Sigma$ is a s.c.c. then it is possible to "cut Σ along γ" to obtain one (or two) new surface(s)

γ is non-separating γ is separating



Rk Cutting along a curve preserves the Euler characteristic

$$\boxed{\chi(\Sigma) = \chi(\Sigma \text{ cut along } \gamma)} = \chi(\Sigma_1) + \chi(\Sigma_2) \quad \text{if } \gamma \text{ is separating}$$

and Σ_1, Σ_2 are the resulting components

This is easy to see assuming that Σ is triangulated and that γ is a union of edges: When cutting along γ the number of triangles remains the same, and we add the same number of edges and vertices this is because we only deal with oriented surfaces.

Vice versa, we can "glue" two compatibly oriented boundary components of one (or more) surface(s) to obtain a new surface

Fact Cutting and glueing are well-defined operations and they are the inverse of one another (in particular, glueing preserves as well the Euler Char.)

There are two ways of formalizing these operations:

Directly: let Σ cut along γ be $(\Sigma \setminus \gamma) \cup \{\text{two copies of } \gamma\}$

and let Σ_1 glue Σ_2 be $\Sigma_1 \amalg \Sigma_2 /\sim$ on identifying points in the boundary

Advantage: the operations are topologically well defined

Disadvantage: need to check by hand that the outcomes are surfaces

(for this we need to parametrize the boundary curves, and then check that the end result does not depend on the param. up to homeo)

Using Collars:

Fact every curve γ has a tubular neighbourhood $N(\gamma)$ (a neighbourhood homeomorphic to $S^1 \times (0,1)$) that is essentially unique up to homotopy

let Σ cut along γ be $\Sigma \setminus N(\gamma)$ (this is already cpt)

define Σ_1 glue Σ_2 by identifying tubular neighbourhoods of the boundary curves

Advantage: the outcomes are clearly surfaces

Disadvantage: need to prove the existence of tubular neighbourhoods and that the outcome does not depend on its choice.

If one is only interested on topological curves and surfaces, the direct approach is probably more efficient. On the other hand, one may want to consider differentiable curves & surfaces, using collars it is easier to deal with smooth structures.

Prop up to homeomorphism there are only finitely many types of s.c.c. in a surface $\Sigma_{g,b}$. Moreover, there exist only one non-separating s.c.c.

proof: let's show that $\exists!$ non-separating s.c.c.

That is, we want to show that if γ_1, γ_2 are non-separating s.c.c. in $\Sigma_{g,b}$, then $\exists$ homeo $\varphi: \Sigma_{g,b} \to \Sigma_{g,b}$ such that $\varphi(\gamma_1) = \gamma_2$

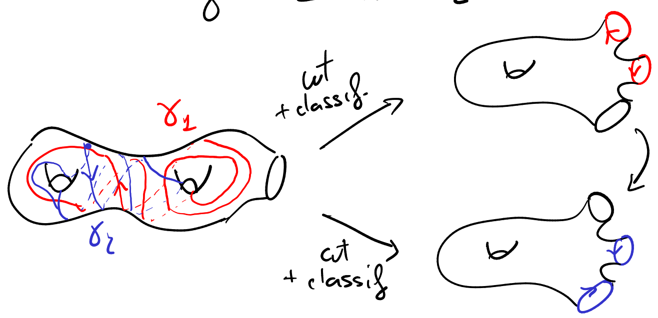
Cut $\Sigma_{g,b}$ along γ_1. By hypothesis, we get a connected surface Σ'.

We know Σ' has $b+2$ boundary components and $\chi(\Sigma') = \chi(\Sigma_{g,b})$

Topological classification $\Rightarrow$ $\boxed{\Sigma' \cong \Sigma_{g-1,b+2}}$

The same argument works for $\Sigma'' = \Sigma_{g,b}$ cut along γ_2.

$\rightsquigarrow$ can find a homeo $\Sigma' \xrightarrow{\varphi} \Sigma''$ that induces an homeo $\bar{\varphi}: \Sigma_{g,b} \circlearrowleft$ sending γ_1 to γ_2

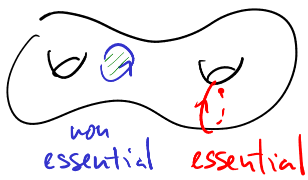


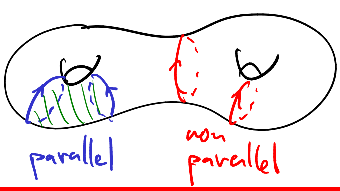
$\rightsquigarrow$ to make sure that φ glues nicely on $\bar{\varphi}$ we also need to match the orientations

The proof of the statement that $\exists$ only finitely many s.c.c. follows the same lines. $\square$

Def a s.c.c. $\gamma \subset \Sigma$ is non-essential if we obtain a disc when cutting Σ along γ (i.e. γ bounds a disc)

Two s.c.c. γ_1 and γ_2 are parallel if they are disjoint and we obtain a cylinder when cutting Σ along γ_1 and γ_2 (they bound a cylinder)





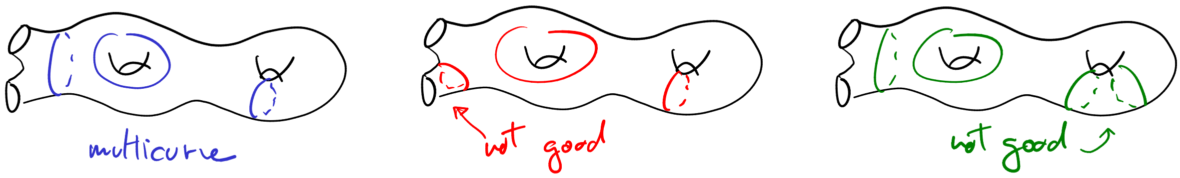
Fact a s.c.c. is non-essential iff it is homotopically trivial

Two disjoint s.c.c. are parallel iff they are freely homotopic

homotopies are not required to fix any base-point.

§2: Multicurves and Pants decompositions

Def a multicurve in Σ is a collection of disjoint essential simple closed curves that are not parallel to one another nor to the boundary components



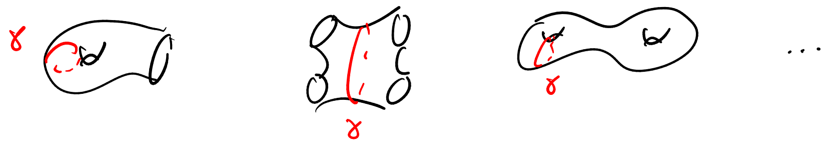
If $(\gamma_1, \dots, \gamma_n) \subset \Sigma$ is a multicurve, we can cut Σ along it by cutting along all the curves γ_i (the order of cutting doesn't matter)

We say that a multicurve is maximal if it is not contained in any strictly larger multicurve

We call the surface $\Sigma_{0,3}$ a pair of pants

Prop Assume $\chi(\Sigma_{g,b}) < 0$ (i.e. $\Sigma_{g,b} \neq S^2, D, S^1 \times S^1, S^1 \times [0,1]$)
and let $\mu = (\gamma_1, \dots, \gamma_n) \subset \Sigma_{g,b}$ be a multicurve.
TFAE: 1) μ is maximal
2) Cutting $\Sigma_{g,b}$ along μ yields a collection of pants
3) $n = 3g + b - 3$

Pf: 1) $\Rightarrow$ 2) Cut $\Sigma_{g,b}$ along μ $\leadsto$ obtain a disjoint union $\Sigma' \amalg \dots \amalg \Sigma^{(k)}$
Each $\Sigma^{(i)}$ has at least one boundary component. None of them can be a disc or an annulus by the definition of multicurve.
If $\Sigma^{(i)}$ is not a pair of pants, it follows from the classification of surfaces that I can find a s.c.c. γ that makes μ into a larger multicurve ↯



2) $\Rightarrow$ 1): if γ is a curve disjoint from μ, then it must be contained in a pair of pants

- $\Sigma_{0,3}$ has no genus $\Rightarrow$ γ must be separating
- $\Sigma_{0,3} = \Sigma'$ glue Σ'' $\Rightarrow$ one among Σ' and Σ'' must be a disc or an annulus because $\chi(\Sigma') + \chi(\Sigma'') = \chi(\Sigma_{0,3}) = -1$

$\Rightarrow$ $\mu \cup \gamma$ is not a multicurve

2) $\Rightarrow$ 3) let Σ cut along μ be the union $\Sigma' \sqcup \dots \sqcup \Sigma^{(k)}$

The total number of boundary components is

$$b + 2n = \sum_i \# \text{ comp. of } \partial\Sigma^{(i)} = 3k \quad \leftarrow n = |\mu|$$

Further, $2-2g-b = \chi(\Sigma_{g,b}) = \sum_{i=1}^{k} \chi(\Sigma^{(i)}) = -k$ and

hence $n = 3g-3+b$

3) $\Rightarrow$ 1) if $n = 3g+b-3$, then μ must be maximal, otherwise it would be contained in some maximal multicurve with too many curves. $\square$

To show that every multicurve is contained in a maximal multicurve it is enough to find a bound on $|\mu| = n$.

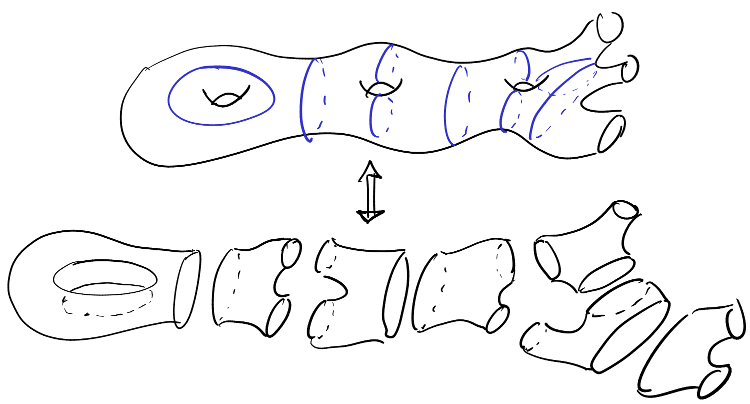
If μ is a multicurve and $(\Sigma \text{ cut along } \mu) = \Sigma^{(1)} \sqcup \dots \sqcup \Sigma^{(k)}$, then

$$\underbrace{-\chi(\Sigma)}_{2g+b-2} = \sum_{i=1}^{k} -\chi(\Sigma^{(i)}) = \sum_{i=1}^{k} 2\,\text{genus}(\Sigma^{(i)}) + \underbrace{\sum_{i=1}^{k} \#(\text{comp. of } \partial\Sigma^{(i)})}_{b+2n} - 2k \quad n = |\mu|$$

$$\Rightarrow n \le g + k - 1$$

On the other hand, $\chi(\Sigma^{(i)}) \le -1 \ \forall i \Rightarrow k \le -\chi(\Sigma)$ is bounded

Bottom line: maximal multicurves give pants decompositions of the surface:

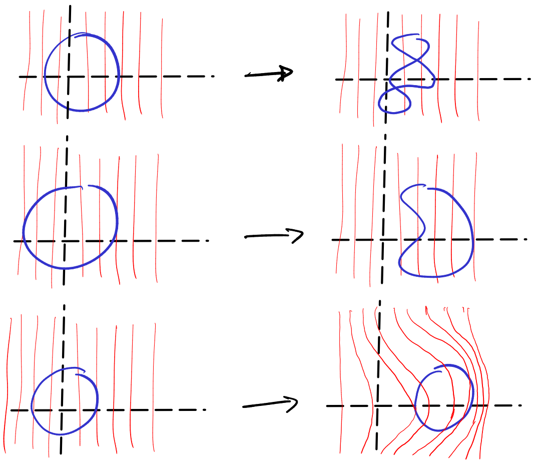


§3: Homotopy & isotopy

Recall: two continuous maps $\varphi_0, \varphi_1 : X \to Y$ are homotopic if $\exists\ H : X\times[0,1] \to Y$ cts s.t. $\varphi_0 = H(\cdot,0)$ and $\varphi_1 = H(\cdot,1)$

Def two embeddings (i.e. that give an homeo with the image) $\varphi_0,\varphi_1 : X \to Y$ are isotopic if they are homotopic via an homotopy H such that $H(\cdot,t): X \to Y$ is an embedding $\forall t\in[0,1]$.

They are ambient isotopic if there exists $\tilde H : Y\times[0,1] \to Y$ s.t. $\tilde H(\cdot,0) = id$, $\tilde H(\cdot,t): Y \to Y$ is a homeo $\forall t$ and $\varphi_1(x) = \tilde H(\varphi_0(x),1)$ (i.e. $\tilde H(\varphi_0(\cdot),\cdot)$ gives an isotopy between φ_0 and φ_1)



homotopy

isotopy

ambient isotopy

Fact 1) if $\mu = (\gamma_1,\dots,\gamma_n)$ and $\mu' = (\gamma_1',\dots,\gamma_n')$ are homotopic multicurves (i.e. $\gamma_1\sim\gamma_1'$, $\gamma_2\sim\gamma_2'$ etc...) then they are ambient isotopic

2) if $\varphi,\varphi' : \Sigma \circlearrowleft$ are homotopic homeomorphisms, then they are isotopic

Rmk one can also talk about smooth (ambient) isotopies between smooth embeddings. I.e. (ambient) isotopies that are also required to be C^∞. The Fact remains true in this setting as well (homotopic diffeomorpisms (smooth multicurves) are smoothly (ambient) isotopic)
This is a low-dimensional phenomenon, that fails in high dimension

§4: Mapping Class Group

Let $\boxed{\mathrm{Homeo}^+(\Sigma, \partial\Sigma)}$ be the group of orient. pres. homeomorphisms $\varphi: \Sigma \circlearrowleft$ that fix the boundary pointwise: $\varphi|_{\partial\Sigma} = \mathrm{id}$

Def: the mapping class group of a surface Σ is the group

$$MCG(\Sigma) := \mathrm{Homeo}^+(\Sigma, \partial\Sigma) / \text{isotopy fixing } \partial\Sigma$$

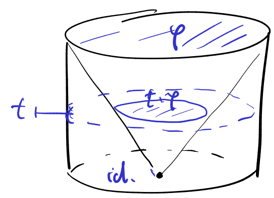
i.e. $H(\cdot, t)|_{\partial\Sigma} = \mathrm{id} \quad \forall t \in [0,1]$

Lemma $MCG(D) = \{e\}$ $\leftarrow$ $D = \{x \in \mathbb{R}^2 \mid |x| \le 1\}$

pf: (Alexander Trick) let $\varphi: D \to D$ be a homeo s.t. $\varphi|_{\partial D} = \mathrm{id}$.

the map $H: D \times I \longmapsto D$

$$(x, t) \longmapsto \begin{cases} t\varphi(\frac{x}{t}) & \text{if } |x| < t \\ x & \end{cases}$$



is a continuous isotopy between id and φ $\square$

Cor $MCG(S^2) = \{e\}$

pf: $\varphi: S^2 \circlearrowleft$ homeo sends a largest circle γ to some s.c.c $\varphi(\gamma)$.

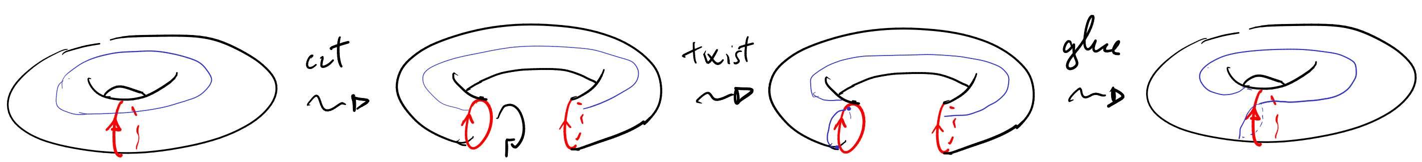
γ and $\varphi(\gamma)$ are homotopic $\Rightarrow$ are ambient isotopic.

$\Rightarrow$ Up to isotopy φ fixes the largest circle

Apply the lemma to both hemispheres $\Rightarrow \varphi$ is isotopic to identity $\square$

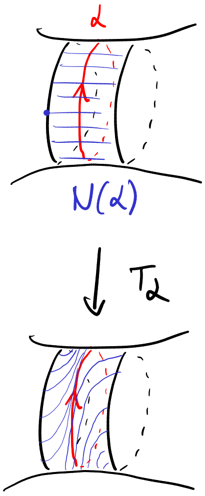
Ok, how do we construct a non-trivial element of some mapping class group?

idea: choose a simple closed curve $\alpha \subset \Sigma$, cut Σ along α, make a twist of angle 2π and glue back Σ:



Formally: let $\alpha : S^1 \to \Sigma$ be a parametrized s.c.c.
Then there exists a "tubular neighbourhood" $N(\alpha) \subset \Sigma$, i.e. for which α extends to a orientation-preserving homeomorphism

$$\boxed{F : S^1 \times (-1, 1) \longrightarrow N(\alpha) \quad \text{s.t.} \quad F|_{S^1 \times \{0\}} = \alpha}$$



define
$$\boxed{\begin{aligned} T : S^1 \times (-1,1) &\longrightarrow S^1 \times (-1,1) \\ (e^{i\theta}, t) &\longmapsto (e^{i\theta + \pi(t+1)}, t) \end{aligned}}$$

and $T_\alpha : \Sigma \to \Sigma$ by

$$\boxed{T_\alpha(x) := \begin{cases} F \circ T \circ F^{-1} & \text{if } x \in N(\alpha) \\ \mathrm{id} & \text{otherwise} \end{cases}}$$

The homeomorphism T_α is called Dehn Twist about α.

Fact Up to isotopy, the definition of Dehn-Twist does not depend on the choice of tubular neighbourhood.

Rmk we call T_α a homeomorphism, but it really is an element of MCG (i.e. it's defined up to isotopy). Note that T_α only depends on the (oriented) homotopy class of α.

The following are two fundamental results that we don't have time to prove:

Th (Dehn-Lickorish) The mapping class group $MCG(\Sigma)$ is generated by finitely many Dehn-Twists

Th (Dehn-Nielsen-Baer) $MCG \cong Out(\pi_1(\Sigma))$

$Out(G)$ = group of outer-automorphisms of G is the quotient of $Aut(G)$ by the group of inner automorphisms (i.e. automorphisms arising as conjugation by some element of G)

§5: Hyperbolic Surfaces

Analogously to the theory of holomorphic coverings, the universal cover $\tilde{\Sigma}$ of a Riemannian surface Σ admits a unique Riemannian metric such that the covering map is a local isometry (this is a Riemannian covering)

With respect to this metric, the group of deck transform $\mathrm{Aut}(\tilde{\Sigma} \to \Sigma)$ acts by isometries on $\tilde{\Sigma}$ (this was left as an Exercise after Lesson 5)

Remember that the Poincaré Disk $\left(D, \left(\frac{2}{1-(x^2+y^2)}\right)^2 \langle\cdot,\cdot\rangle_{Eucl}\right)$ is a surface of constant curvature $K \equiv -1$, and that it is isometric to the Upper Half Plane $\left(H, \frac{1}{y^2}\langle\cdot,\cdot\rangle_{Eucl}\right)$. This surface is the Hyperbolic plane $\mathbb{H}^2$

Def A closed Riemannian surface is hyperbolic if its universal cover is $\mathbb{H}^2$

Fact this is equivalent to saying that Σ has constant curvature $K \equiv -1$

(one implication is clear. The other implication follows from the fact that $\mathbb{H}^2$ is the unique simply connected surface of constant curvature $K \equiv -1$)

orientation preserving Riem. isometries

bi-holomorphisms ($PSL(2,\mathbb{R})$)

Lemma $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{Aut}(D)$

Pf:
- $\mathrm{Isom}^+(\mathbb{H}^2) \leq \mathrm{Aut}(D)$ because isometries are conformal equivalences.
- It remains to show that every bilolom. $\varphi \in \mathrm{Aut}(D)$ is an isometry.

  $\mathrm{Isom}^+(\mathbb{H}^2)$ acts transitively on $\mathbb{H}^2$ (to prove this it's enough to note that multiplication by scalar is an isometry of H and that rotations are isometries of D). $\Rightarrow$ WLOG $\varphi(0) = 0$.

  It then follows from Schwarz Lemma (Chapter 3 Sec 3) that φ is a rotation and hence an isometry. $\square$

i.e. a Riemannian metric making it hyperbolic

Cor/Ex A closed surface Σ_g admits a hyperbolic structure iff $g \geq 2$.
Furthermore, in this case there is a natural correspondence

$$\{\text{hyperbolic structures}\} \xleftrightarrow{1-1} \text{Moduli Space}(\Sigma_g)$$

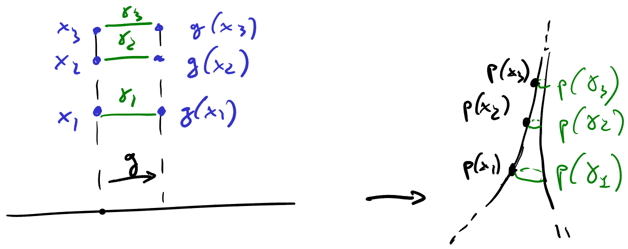
Since Isometries of $\mathbb{D}$ are Möbius transformations, we know that they can be subdivided into Elliptic, Parabolic and Hyperbolic transformations.

Fact If Σ is a compact Riemannian surface, there is an $\varepsilon > 0$ such that every curve of length $\leq \varepsilon$ is homotopically trivial

Cor If Σ is a closed hyperbolic surface and $p : \mathbb{H}^2 \to \Sigma$ the universal cover, then $\mathrm{Aut}(\mathbb{H}^2 \to \Sigma)$ consists of hyperbolic isometries

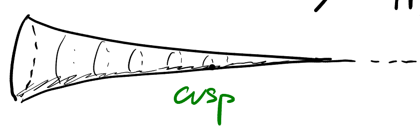
Pf: We know that $\mathrm{Aut}(\mathbb{H}^2 \to \Sigma)$ is a group of isometries that don't have fixed points $\Rightarrow$ it's enough to show that it does not contain parabolics

By contradiction, let $g \in \mathrm{Aut}(\mathbb{H}^2 \xrightarrow{p} \Sigma)$ be parabolic. We can assume that it is a translation in the Half-Plane model. Choose points $x_1, x_2, \dots$ that go to ∞ and pick curves γ_i joining x_i to $g(x_i)$. Their projection $p(\gamma_i)$



are closed curves in Σ that become arbitrarily short but are not homotopically trivial. This gives a contradiction with cptness of Σ $\square$

Rmk More precisely, the proof showed that there is a correspondence between parabolic elements in $\mathrm{Aut}(\mathbb{H}^2 \to \Sigma)$ and cusps in Σ. We are working with closed hyperbolic surfaces for convenience: one is usually happy to work with hyperbolic surfaces with cusps.



Recall that the Gauss-Bonnet formula relates the integral of the curvature on a geodesic triangle with the sum of the internal angles.

With some extra work it is possible to prove that
$$\boxed{\int_\Sigma K(p)\, dArea = 2\pi\, \chi(\Sigma)}$$
Euler Char.

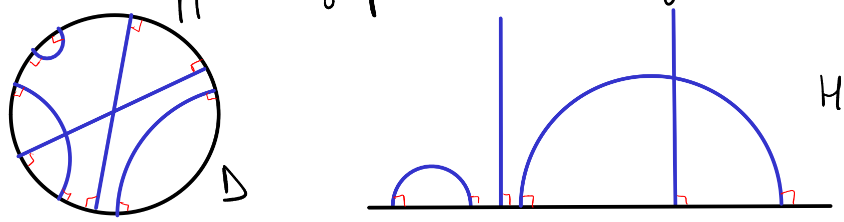
(this was left as an exercise after lec. 4)

Cor if Σ_g is a hyperbolic surface, its area is equal to $2\pi(2g-2)$ (in particular, it does not depend on the specific choice of hyperbolic metric)

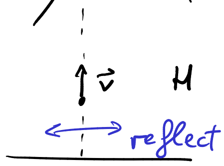
§6: Curves & Geodesics

we ignore the parametrization because for the moment we will not need it.

Lemma Geodesics in $\mathbb{D}$ are precisely those lines and circles that are orthogonal to $\partial\mathbb{D}$. The same characterization holds for the geodesic in the upper half-plane model for $\mathbb{H}^2$



proof: in H the vertical lines are geodesics. You can either check it by hand (easy) or say as follows: We know that $\exists!$ geodesic γ starting with velocity $\vec{v}$. Reflecting across the dotted line is a (orientation reversing) Riemannian isometry that fixes $\vec{v}$ $\Rightarrow$ it must fix the geodesic through v $\Rightarrow$ γ must be the dotted line



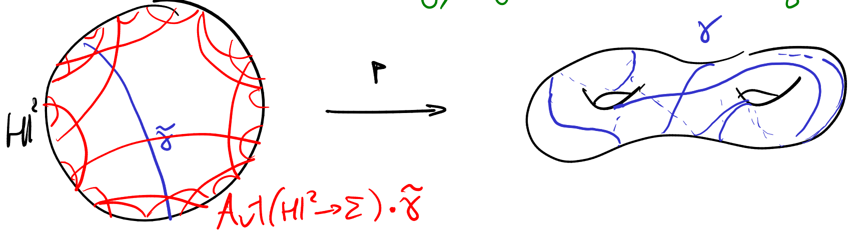
All the other lines and circles perpendicular to ∂H can be mapped to γ using some Möbius transf $\Rightarrow$ they're all geodesics

For every point $x \in H$ and $\vec{v} \in T_x H$ there is such a line or circle passing through x with direction $\vec{v}$. By uniqueness of geodesics we deduce that every geodesic is such a line or circle $\square$

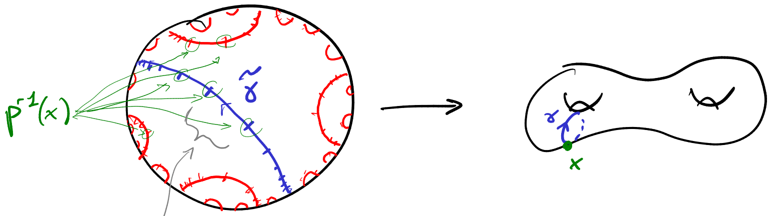
Cor for any two distinct points $x, y \in \mathbb{H}^2$ $\exists!$ geodesic connecting them.

Let $p: \mathbb{H}^2 \to \Sigma$ be a hyperbolic surface and $\gamma: \mathbb{R} \to \Sigma$ a geodesic since $\mathbb{R}$ is simply connected, the map $\gamma: \mathbb{R} \to \Sigma$ lifts to maps $\tilde{\gamma}: \mathbb{R} \to \mathbb{H}^2$ which are geodesics because p is a local isometry. It follows that $\boxed{p^{-1}(\gamma) \text{ is a } \mathrm{Aut}(\mathbb{H}^2 \to \Sigma)\text{-invariant collection of geodesics}}$

we are identifying γ with its image



If $\gamma: S^1 \to \Sigma$ is a closed geodesic, we can see it as a periodic infinite geodesic: $\mathbb{R} \to S^1 \to \Sigma$ $\rightsquigarrow$ we can lift this to $\mathbb{H}^2$



this segment can be seen as the lift obtained by "walking once along γ"
$\tilde{\gamma}$ is obtained by walking infinitely many times along γ.

Rule the pre-image $p^{-1}(\gamma)$ is a disjoint collection of curves if and only if the image of $\gamma: \mathbb{R} \to \Sigma$ is a simple curve.
More about this in the exercises...

Prop if Σ is a hyp. surface and $\gamma: S^1 \to \Sigma$ is any closed curve, then $\exists!$ closed geodesic $\bar{\gamma}: S^1 \to \Sigma$ that is (freely) homotopic to γ

proof: let $\gamma_n : S^1 \to \Sigma$ be a sequence of closed curves s.t.

- $\gamma_n \sim \gamma \quad \forall n$
- $\|\gamma_n\| \xrightarrow{n\to\infty} \inf \{ \|\eta\| \mid \eta \sim \gamma \}$

(free) homotopy

length

As functions among metric spaces, γ_n are equicontinuous

(Well, technically we need to choose the γ_n to "move at constant speed", i.e. we want parametrizations such that $\|\gamma_n|_{(0,\vartheta)}\| = \frac{\vartheta}{2\pi}\|\gamma_n\|$)

Since Σ is compact, it follows by the Ascoli-Arzelà Theorem that there exists a subsequence γ_{n_k} that converges uniformly to some continuous curve $\boxed{\bar{\gamma}: S^1 \to \Sigma}$.
It is easy to verify that $\boxed{\bar{\gamma} \sim \gamma}$ and $\boxed{\|\bar{\gamma}\| = \inf \{\|\eta\| \mid \eta \sim \gamma\}}$.
It follows that $\bar{\gamma}$ is a geodesic (it minimizes the length)
(in particular it's also nice & smooth).

Remains to see that $\bar{\gamma}$ is the unique geodesic homotopic to γ

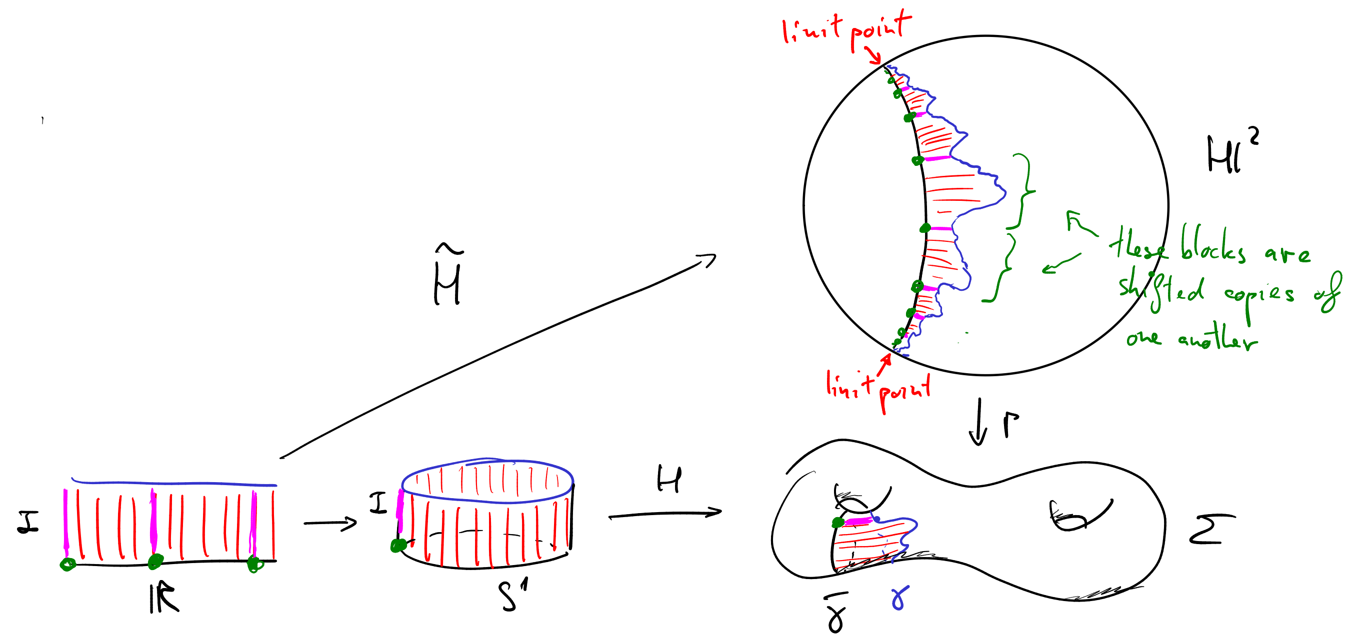
Say that $H: S^1 \times I \longrightarrow \Sigma$ is a homotopy between γ and $\bar{\gamma}$.
Choose a lift $\tilde{H}$ to the universal cover:

$$\begin{tikzcd} & & \mathbb{H}^2 \arrow[d, "p"] \\ \mathbb{R} \times I \arrow[r] \arrow[urr, dashed, "\tilde{H}"] & S^1 \times I \arrow[r, "H"'] & \Sigma \end{tikzcd}$$

(the functions $\tilde{H}(-,0)$ and $\tilde{H}(-,1)$ correspond to a bi-infinite lift of γ and $\bar{\gamma}$ respectively)

Since $\tilde{H}(-,0)$ is the lift of $\bar{\gamma}$, it is a bi-infinite geodesic.
Furthermore, $\tilde{H}$ is "periodic" and hence $\tilde{H}(-,1)$ is at bounded distance from $\tilde{H}(-,0)$

On the level of maps to the universal cover, doing a whole revolution around S^1 corresponds to shifting everything by a deck transformation



In particular, it follows that $\tilde{H}(-,1)$ "converges" to the same limit points in $\partial\mathbb{H}$ as does the lift of $\bar{\gamma}$.
Furthermore, any geodesic that is homotopic to γ must converge to the same limit points. But we know that $\exists!$ geodesic in $\mathbb{H}^2$ passing through them, and thus we're done

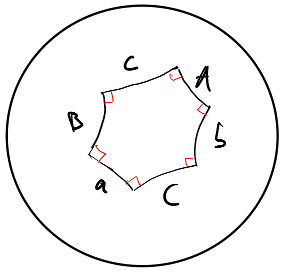
$\square$

Cor/Ex Any choice of hyperbolic structure on Σ induces a well-defined function $\boxed{\begin{matrix} \pi_1(\Sigma) \longrightarrow \mathbb{R}_{>0} \\ \gamma \longmapsto |\gamma| \end{matrix}}$ that is invariant under conjugation.

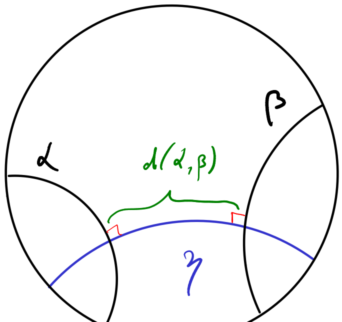
§7: Constructing hyperbolic metrics

We know by our work on Riemann surfaces that every surface Σ_g with $g \geq 2$ can be given a hyperbolic metric. We now want to construct hyperbolic metrics more explicitely.

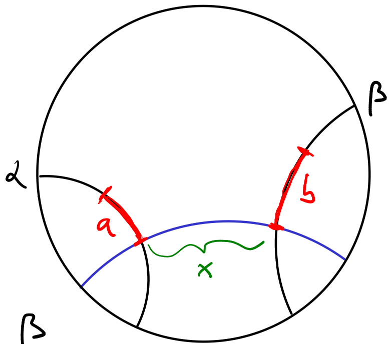
Lemma given $a, b, c > 0$, there exist unique $A, B, C > 0$ and a right-angled geodesic hexagon in $\mathbb{H}^2$ with edges of length a, C, b, A, c, B (such hexagon is unique up to isometry)



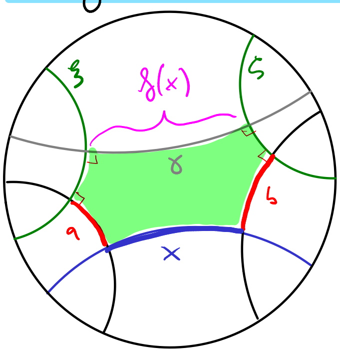
pf: Note that for any two distinct geodesics α, β in $\mathbb{H}^2$ $\exists$ geodesic η perpendicular to both of them if and only if $0 < d(\alpha, \beta) = \inf\{d(x,y) \mid x \in \alpha, y \in \beta\}$. When that is the case, η is unique and the length of the segment of η between α and β is precisely $d(\alpha, \beta)$ (i.e. η realizes the distance between them)



Pick segments of length $\boxed{a}$ and $\boxed{b}$ on α, β starting from the intersection with η.
Let $x > 0$ be the distance between them



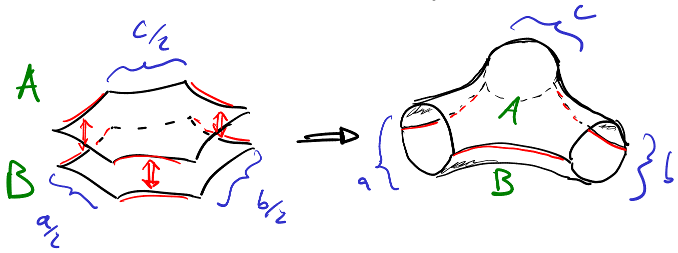
$\exists!$ geodesics ξ and ζ perpendicularly meeting α and β at the extremities of such segments and then there exists (at most) one geodesic γ that is perpendicular to both of them.
That is, $\boxed{\text{if}}$ there exists a right-angled geodesic hexagon with edges $\boxed{a, x, b}$ then it's unique.



Let $\boxed{\delta(x)}$ be the distance between the curves ξ and ζ, seen as a function of x. There is some value x_0 s.t. $\delta(x_0) = 0$ and $\delta(x) > 0$ $\forall x > x_0$.
You can check that δ is strictly monotone for $x > x_0$ and that $\delta(x) \xrightarrow{x \to \infty} \infty$. By continuity, $\exists!\, x$ s.t. $\delta(x) = c$ $\square$

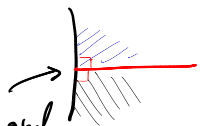
Prop $\forall\, a,b,c>0\ \exists!$ hyperbolic pair of pants (unique up to isometry) with geodesic boundary whose boundary components have length a,b,c

Pf: Existence: let A, B be two copies of the right-angled hexagon with lengths $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ and glue them to obtain a pair of pants



- Since we are gluing hexagons along geodesics of the same length, we naturally obtain a hyperbolic metric on the pair of pants

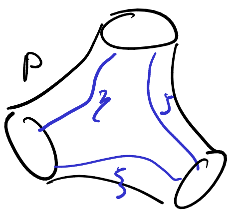
- Since the hexagons are right-angled, the boundary components of the pants are smooth geodesics (and have the correct length)



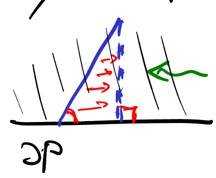
2 right angles make for a smooth curve

Uniqueness: let $\boxed{P}$ be a pair of pants with hyperbolic metric and geodesic boundary components of length a,b,c.

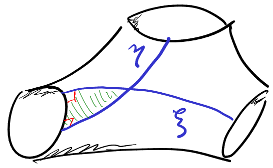
let $\boxed{\eta, \xi, \zeta}$ be paths realizing the distance between the boundary components



Since they minimize the distance, they must be geodesics. Moreover, they must be perpendicular to the boundary components otherwise the could be shortened by "sliding"
(this is a computation of Riemannian geometry)



η, ξ, ζ are disjoint because otherwise we would find a hyperbolic geodesic triangle with internal angles adding up to π or more, against Gauss-Bonnet



It follows that if we cut P along η, ξ, ζ we obtain two right-angled geodesic hexagons with edges of length $|\eta|, |\xi|, |\zeta|$ (this can be checked using Euler's Characteristic)

$\Rightarrow$ those two hexagon must be isometric and each of them must have edges of length $a/2, b/2, c/2$

That is, the metric on X is obtained precisely via the glueing construction that we did before (there was no other choice) $\square$

This proposition can be used to construct a lot of concrete hyperbolic metrics on any surface Σ_g: choose a pants decomposition (i.e. a maximal multicurve $\mu = (\gamma_1, \dots, \gamma_{3g-3})$) and choose a positive number $a_i > 0$ $\forall i = 1, \dots, 3g-3$. On each pair of pants in the decomposition, there is a unique hyperbolic metric such that the boundary components are geodesics of length a_i (for the appropriate i's). When we put the pants together we obtain a hyperbolic metric on Σ_g where the curve γ_i is a geodesic of length a_i