Chapter 3: Riemann Surfaces

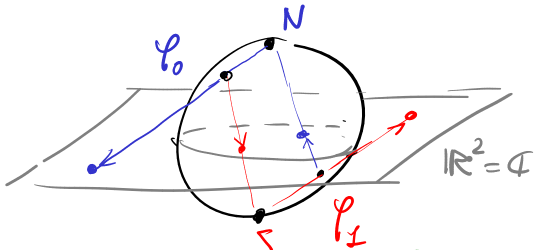
In this chapter all surfaces will have no boundary!

§1: Uniformization

homeomorphisms that are locally bi-holomorphic

Recall:
- We care about Riemann Surfaces up to conformal equivalences
- The Riemann mapping theorem says that every proper domain $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to the unit disk $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ (in books they often call this U or B)
- $\mathbb{D}$ and $\mathbb{C}$ are not conformally equivalent

The sphere $\mathbb{S}^2$ has a natural complex structure. It can either be seen as $\mathbb{C} \cup \{\infty\}$ or, more similarly to the way we defined Riemann surfaces, one can describe its complex structure with an atlas with two charts
$$U_0 = \mathbb{S}^2 \setminus \{N\}$$
$$U_1 = \mathbb{S}^2 \setminus \{S\}$$



if $\varphi_i : U_i \to \mathbb{C}$ is the diffeomorph. given by the stereographic projection, then $\varphi_i(U_i \cap U_j) = \mathbb{C} \setminus \{0\}$ and the change-of-coordinate is given by $\frac{1}{z}$

this is anti-holomorphic

To get a proper complex structure we post-compose φ_1 with a reflection $\varphi_1'(p) := \overline{\varphi_1(p)}$

complex-conjugate

Of course, $\mathbb{S}^2$ is not conformally equivalent to $\mathbb{D}$ nor $\mathbb{C}$ because they are not homeomorphic.

The following is a fundamental result in the theory of Riemann Surfaces:

Thm (Riemann Uniformization) If Σ is a simply-connected Riemann surface then it is conformally equivalent to one of $\mathbb{D}, \mathbb{C}, \mathbb{S}^2$

Rmk If we knew that every Riemann surface can be conformally embedded into $\mathbb{C}$, this theorem would follow from the Riemann Mapping Theorem.
It is indeed possible to prove the Theorem this way, but it's far from trivial.

Recall that every topological surface admits a complex structure. Then we get:

Cor every simply connected topological surface is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$

To classify the other surfaces we need to use some theory of coverings.

Recall that every (topological) surface Σ admits a universal cover $\pi: \tilde{\Sigma} \longrightarrow \Sigma$ where $\tilde{\Sigma}$ is a simply-connected surface.

Σ is the quotient of $\tilde{\Sigma}$ by the group of deck-transformations.
Moreover, this group is isomorphic to $\pi_1(\Sigma)$

$$\underbrace{\mathrm{Aut}(\tilde{\Sigma} \xrightarrow{\pi} \Sigma)} := \left\{ \psi: \tilde{\Sigma} \to \tilde{\Sigma} \;\middle|\; \begin{tikzcd} \tilde{\Sigma} \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & \tilde{\Sigma} \arrow[dl, "\pi"] \\ & \Sigma & \end{tikzcd} \text{ commutes} \right\} \cong \pi_1(\Sigma)$$

Deck transformations

Lemma if Σ is a Riemann surface then $\tilde{\Sigma}$ admits a natural complex structure s.t. π is holomorphic. Furthermore, $g: \tilde{\Sigma} \to \tilde{\Sigma}$ is bi-holomorphic for every $g \in Aut(\tilde{\Sigma} \to \Sigma)$

pf: let $\{\varphi_i : V_i \to \mathbb{C}\}$ be the complex atlas of Σ. shrinking and subdividing the V_i if necessary, we can assume that they are small enough s.t.

$\boxed{\pi^{-1}(V_i) = \coprod_g g(\tilde{V}_i)}$ where $\tilde{V}_i$ is a (fixed) connected component of $\pi^{-1}(V_i)$ and g ranges in $Aut(\tilde{\Sigma} \to \Sigma)$

In particular $\pi : g(\tilde{V}_i) \to V_i$ is a diffeo and we can let $\boxed{\pi \circ \varphi_i : g(\tilde{V}_i) \to \mathbb{C}}$ be the charts on $\tilde{\Sigma}$. Finishing the proof is an exercise $\square$

Rmk vice versa, one can show that is Σ is a Riemann surface and $\Gamma < Aut(\Sigma)$ is a group of bi-holomorphisms such that the action $\Gamma \curvearrowright \Sigma$ is free and properly discontinuous, then the quotient Σ/Γ admits a natural structure of Riemann surface such that $\pi : \Sigma \to \Sigma/\Gamma$ is holomorphic.

We call this kind of coverings holomorphic coverings.

Rmk if $\pi_1 : \overline{\Sigma}_1 \to \Sigma_1$ and $\pi_2 : \overline{\Sigma}_2 \to \Sigma_2$ are two holomorphic covering and $F : \Sigma_1 \to \Sigma_2$ is a holomorphic map that admits a lift $\overline{F} : \overline{\Sigma}_1 \to \overline{\Sigma}_2$ (i.e the diagram
$$\begin{tikzcd} \overline{\Sigma}_1 \arrow[r, "\overline{F}"] \arrow[d, "\pi_1"'] & \overline{\Sigma}_2 \arrow[d, "\pi_2"] \\ \Sigma_1 \arrow[r, "F"] & \Sigma_2 \end{tikzcd}$$
commutes)

then $\overline{F}$ is holomorphic

Thm Every Riemann surface Σ is conformally equivalent to X/Γ, where X is either S^2, $\mathbb{C}$ or $\mathbb{D}$, Γ is a group of bi-holomorphism acting freely and properly-discontinuously on X. Furthermore, $\boxed{\Gamma \cong \pi_1(\Sigma)}$

pf: let $\tilde{\Sigma}$ be the universal cover, then $\boxed{\Sigma = \tilde{\Sigma}/\mathrm{Aut}(\tilde{\Sigma}\to\Sigma)}$ and the action is free, prop. disc.

$\mathrm{Aut}(\tilde{\Sigma}\to\Sigma) \cong \pi_1(\Sigma)$

by the Riemann-Uniformization we know that there is a conf. equivalence $\tilde{F}: \tilde{\Sigma} \longrightarrow X$ for an appropriate X.

For every $g \in \mathrm{Aut}(\tilde{\Sigma}\to\Sigma)$, $\tilde{F} g \tilde{F}^{-1}: X \longrightarrow X$ is a conformal eq.
We let $\Gamma := \{\tilde{F} g \tilde{F}^{-1} \mid g \in \mathrm{Aut}(\tilde{\Sigma}\to\Sigma)\}$ and it is now easy to check that

$$\boxed{\begin{array}{l}\tilde{F}: \tilde{\Sigma} \to X \text{ descends to a conformal equivalence} \\ F: \tilde{\Sigma}/\mathrm{Aut}(\tilde{\Sigma}\to\Sigma) \longrightarrow X/\Gamma\end{array}}$$

$\square$

Rmk Note that if Σ_1 and Σ_2 are conf. equiv. then also their universal covers $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are conf. equiv. (any homeo $\Sigma_1 \to \Sigma_2$ lifts to a homeo of the universal covers).

In order to describe non-simply-connected Riemann surfaces we thus need to understand what are the holomorphic automorphisms of $\mathbb{D}$, $\mathbb{C}$, S^2

§2 Automorphisms groups of $\mathbb{C}$ and S^2

here by "automorphism" we mean self-biholomorphisms

$\boxed{\mathbb{C}}$: there are 2 obvious kinds of automorphisms for $\mathbb{C}$.

- complex homotheties $(z \mapsto az)$ for some $a \in \mathbb{C} \setminus \{0\}$
  (these include real homotheties and rotations around the origin)
- translations $(z \mapsto z+b)$ for some $b \in \mathbb{C}$

Thm All the automorphisms of $\mathbb{C}$ are of the form $(z \mapsto az+b)$ for some $a, b \in \mathbb{C}$, $a \neq 0$ (i.e. they are compositions of complex homotheties & translations)

Pf: let $F: \mathbb{C} \to \mathbb{C}$ be a biholomorphism.
Since it is holomorphic everywhere, it has an expansion as power series $\boxed{F(z) = \sum_{n \geq 0} a_n z^n}$. ← this series converges $\forall z \in \mathbb{C}$

Consider the holom function $G: \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$

$$z \mapsto F\left(\tfrac{1}{z}\right)$$

Then G has as Laurent expansion on $\mathbb{C} \setminus \{0\}$

$$\boxed{G(z) = \sum_{n \geq 0} a_n \frac{1}{z^n}}$$

[Rudin, Exercise 10.25]

It is a fact of complex analysis that any holomorphic function $f: \Omega \to \mathbb{C}$ defined on an annulus $\Omega = \{ z \mid r < |z| < R \}$ can be expressed in a unique way as a biinfinite series:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

← n can be negative!

i.e. $\exists$ unique choice of a_n $n \in \mathbb{Z}$ such that this series converges to $f(z)$ $\forall z \in \Omega$.

In our case, $F(z) = \sum_{n \geq 0} a_n z^n \rightsquigarrow G(z) = F\left(\frac{1}{z}\right) = \sum_{n \geq 0} a_n \frac{1}{z^n}$

$$\rightsquigarrow G(z) = \sum_{n \leq 0} a_{-n} z^n$$

← uniquely determined

Since F is a homeomorphism, $F(\{z \mid |z| \geq 1\})$ is not dense in $\mathbb{C}$.
Hence, $\boxed{G(\{z \mid 0 < |z| \leq 1\}) \text{ is not dense in } \mathbb{C}}$.

Weierstrass theorem implies that 0 cannot be an essential singularity for G. I.e., there is N s.t. $a_n = 0 \ \forall n > N$.

$\Rightarrow$ $\boxed{F \text{ is a polynomial}}$!

$\Rightarrow$ it has $\boxed{\deg(F) \text{ roots in } \mathbb{C}}$.

If $p \in \mathbb{C}$ is a multiple root of F, then $F(z-p) = (z-p)^k Q(z)$
which cannot be injective near p. $Q(p) \neq 0$

$\Rightarrow$ $\boxed{\text{All the roots are simple.}}$

$\xrightarrow{F \text{ inj.}}$ $\boxed{\text{only one root}}$. I.e. $F(z) = a_0 + a_1 z$ is linear $\square$

$\boxed{S^2}$: Recall that S^2 has 2 charts, $U_0 = S^2 \setminus \{N\} \to \mathbb{C}$
$U_1 = S^2 \setminus \{S\} \to \mathbb{C}$

and that the change-of-coordinate is $\boxed{\begin{array}{c} \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\} \\ z \mapsto 1/z \end{array}}$

Lemma every automorph. of U_0 extends to an automorphism of S^2

Pf:

let

$$\begin{array}{ccc} U_0 & \xrightarrow{F} & U_0 \\ \varphi_0 \downarrow \cong & & \cong \downarrow \varphi_0 \\ \mathbb{C} & \xrightarrow{F_0} & \mathbb{C} \end{array}$$

be biholomorph. We need to show that it extends at $N \in S^2$ remaining a biholomorphism.

Let $\Omega_R := \{z \mid |z| > R\}$. Since $F_0 : \mathbb{C} \to \mathbb{C}$ is a homeomorphism there exists $R \gg 0$ s.t. $F_0(z) > 1 \quad \forall z \in \Omega_R$.

$\boxed{\begin{array}{l}\text{In particular, both } \Omega_R \text{ and } F_0(\Omega_R) \text{ are contained in } \varphi_0(U_0 \cap U_1). \\ \text{We can therefore look at the restriction } F_0|_{\Omega_R} \text{ in the other chart.}\end{array}}$

$$\begin{tikzcd}
& F_0 : \Omega_R \arrow[r, "\simeq"] & F_0(\Omega_R) & \\
S^2 \setminus \{N,S\} = U_0 \cap U_1 \supseteq \arrow[d, "\varphi_1"] & \varphi_0^{-1}(\Omega_R) \arrow[u, "\varphi_0"] \arrow[r, "\simeq"] \arrow[d, "\varphi_1"] & \varphi_0^{-1}(F_0(\Omega_R)) \arrow[u, "\varphi_0"'] \arrow[d, "\varphi_1"'] & \subseteq U_0 \cap U_1 = S^2 \setminus \{N,S\} \arrow[d, "\varphi_1"'] \\
\mathbb{C} \setminus \{0\} \supseteq & \underbrace{\varphi_1 \circ \varphi_0^{-1}(\Omega_R)} \arrow[r, "\simeq", "F_1"'] & \underbrace{\varphi_1 \circ \varphi_0^{-1}(F_0(\Omega_R))} & \subseteq \mathbb{C} \setminus \{0\}
\end{tikzcd}$$

this is the change-of-coordinate $\psi : z \mapsto \frac{1}{z}$

$\leadsto$ we have that $\psi(\Omega_R) = \{ z \mid 0 < |z| < \frac{1}{R} \}$
and that $F_1(z) < 1$ for every $z \in \psi(\Omega_R)$.

That is, $\boxed{F_1 \text{ is bounded on } \psi(\Omega_R)}$.

Then Complex analysis implies that $\boxed{F_1 \text{ extends to } 0}$
and the extension $F_1 : \{ z \mid |z| < \frac{1}{R} \} \to \mathbb{C}$ is holomorphic.
(the Riemann Extension theorem [Rudin, Thm 10.20])

Note also that if p_n is a sequence of points $p_n \in S^2 \setminus \{N\}$ s.t.
$p_n \to N$ then $|\varphi_0(p_n)| \to \infty$ and also $|F_0(\varphi_0(p_n))| \to \infty$
because F_0 is a proper map.
It follows that the extension of $\boxed{F_1 \text{ sends } 0 \text{ to } 0.}$

By construction, the maps $\varphi_0^{-1} \circ F_0 \circ \varphi_0$ and $\varphi_1^{-1} \circ F_1 \circ \varphi_1 : S^2 \to S^2$
agree on their intersection and hence define a homeomorph.
$\hat{F} : S^2 \to S^2$ that is a bi-holomorphism □

Rmk S^2 is the Hausdorff compactification of U_0. It is hence
clear that $F : U_0 \xrightarrow{\simeq} U_0$ extends to a homeomorph. $\hat{F} : S^2 \to S^2$
The thing that is not clear a priori is that this extension
is holomorphic (e.g., the analogous statement for smooth
maps is false). This is where complex analysis magic happens.

$\boxed{\text{It is convenient to use the chart } U_0 \text{ to identify } S^2 \text{ with } \mathbb{C}\cup\{\infty\}. \ \infty \text{ corresponds to the point } N.}$

The previous Lemma then states that the map

$$\begin{aligned} \mathbb{C}\cup\{\infty\} &\longrightarrow \mathbb{C}\cup\{\infty\} \\ z &\longmapsto az+b \end{aligned} \quad \text{with } a\neq 0 \text{ is a biholomorph.}$$

(with the convention $a\cdot\infty=\infty \ \forall a\neq 0$)

Note also that the map $\boxed{\begin{aligned} \mathbb{C}\setminus\{0\} &\xrightarrow{inv} \mathbb{C}\setminus\{0\} \\ z &\longmapsto 1/z \end{aligned}}$ extends to an autom. of S^2

(we have $U_0\setminus\{S\} \xrightarrow{inv} U_1\setminus\{N\}$ and if we write it in coordinates
$$\begin{tikzcd} U_0\setminus S \arrow[r] \arrow[d, "\varphi_0"] & U_1\setminus\{N\} \arrow[d, "\varphi_1"] \\ \mathbb{C}\setminus\{0\} \arrow[r] & \mathbb{C}\setminus\{0\} \end{tikzcd}$$
it is nothing but $\begin{aligned} \mathbb{C}\setminus\{0\} &\longrightarrow \mathbb{C}\setminus\{0\} \\ z &\longmapsto z \end{aligned}$ which clearly extends at 0.)

Of course, also the automorphism of U_1 extend to S^2

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d, "\varphi_1"] & U_1 \arrow[d, "\varphi_1"] \\ \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$
$z \mapsto zd+c$

induce $\rightsquigarrow$ $c,d\in\mathbb{C}$, $d\neq 0$

$$\boxed{\begin{aligned} \mathbb{C}\cup\{\infty\} &\longrightarrow \mathbb{C}\cup\{\infty\} \\ z &\longmapsto \frac{1}{d\frac{1}{z}+c} = \frac{z}{zc+d} \end{aligned}}$$

Note that $Aut(U_1) < Aut(S^2)$ is obtained by conjugating $Aut(U_0) < Aut(S^2)$ with the inversion: $S^2 \xrightarrow{inv} S^2$ sending $z\mapsto \frac{1}{z}$

Technically, all the writing of the sort $z\mapsto\frac{1}{z}$, $\frac{z}{zc}$ etc are only defined for $z\neq\infty$, when the denominator is not 0 etc. To give it a meaning on these special points it would be necessary to say that the function is extended by continuity on the whole of $\mathbb{C}\cup\{\infty\}$.

Luckily, the convention $S^2=\mathbb{C}\cup\{\infty\}$ is a particularly happy one because the obvious conventions $\frac{a}{\infty}:=0$, $\frac{0}{a}:=\infty$, $\frac{\infty}{a\infty}:=a$ etc. make sense and are indeed correct (at least for the functions that we care about.)

$\mathbb{P}_{\mathbb{C}}$ means complex-projective (quotient out by complex scalars) while $\mathbb{P}$ means real projective. $\mathbb{P}_{\mathbb{C}} Gl(2,\mathbb{C}) \neq \mathbb{P}Gl(2,\mathbb{C})$ but $\mathbb{P}Sl(2,\mathbb{C}) = \mathbb{P}_{\mathbb{C}} Sl(2,\mathbb{C}) \cong Sl(2,\mathbb{C})/_{\pm 1}$

It is now easy to prove the following:

Thm the group of biholomorphisms of S^2 is $\mathbb{P}Sl(2,\mathbb{C}) \simeq \mathbb{P}_{\mathbb{C}}Gl(2,\mathbb{C})$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2,\mathbb{C})$ is sent to $M_A := \left(z \longmapsto \dfrac{az+b}{cz+d} \right)$.

Furthermore, for any two triplets of distinct points (s,p,q) (s',p',q') there is a unique $F \in Aut(S^2)$ st $\boxed{F(s)=s' \quad F(p)=p' \quad F(q)=q'}$.

Proof Note that we already showed that M_A is a biholomorph when $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Gl(2,\mathbb{C})$.

Now there are (at least) two ways to proceed:

Step 1 check by hand that M_A is a bi-holom. of S^2 $\forall A \in Gl(2,\mathbb{C})$

Step 2 check that $M_A \circ M_B = M_{A \circ B}$

Step 1 Check formally that $M_A \circ M_B = M_{A\circ B}$

Step 2 Note that $\left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ c & d \end{pmatrix} \right\}$ generates $Gl(2,\mathbb{C})$ and deduce that M_A is a bi-holom. of S^2 $\forall A \in Gl(2,\mathbb{C})$

Step 3 Note that $Gl(2,\mathbb{C}) \to Aut(S^2)$ quotients through $\mathbb{P}_{\mathbb{C}}Gl(2,\mathbb{C}) = \mathbb{P}Sl(2,\mathbb{C})$

We now have a hom $\boxed{\Phi : \mathbb{P}Sl(2,\mathbb{C}) \longrightarrow Aut(S^2).}$
We need to show it's an isomorphism.

Step 4 check Φ is injective.

Step 5 Show that every triple of distinct points (s,p,q) can be sent to $(0,1,\infty)$

Step 6 Given any $F \in \mathrm{Aut}(\mathbb{S}^2)$, can find $A \in \mathbb{P}SL(2,\mathbb{C})$ s.t. $M_A \circ F(\infty) = \infty$. Use the characterization of $\mathrm{Aut}(\mathbb{C})$ to deduce that Φ is surjective

We are now almost done: we know that Φ is an isom. It only remains to check the last part of the "furthermore".

Step 7: Use the characterization of $\mathrm{Aut}(\mathbb{C})$ to deduce that any $F \in \mathrm{Aut}(\mathbb{S}^2)$ st. $F(\infty)=\infty$ $F(0)=0$ $F(1)=1$ must be the identity.

$\square$

Exercise: fill the details of the pf

Def the maps $M_A : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ are called Möbius transformations.

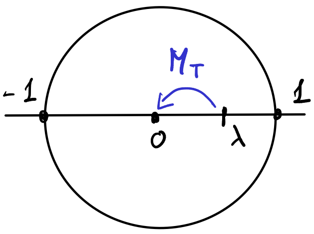
Rmk Note that Möbius transformations send circles to circles in $\mathbb{S}^2$ (or, send circles & lines to circles & lines in $\mathbb{C}$)

§3 Autonorphisms of $\mathbb{D}$

We start by studying Möbius transforms of $\mathbb{S}^2$.

Pick $0 < \lambda < 1$. Then $\exists! \; T \in SL(2,\mathbb{C})$ s.t.

$$\boxed{\begin{cases} M_T(\lambda) = 0 \\ M_T(1) = 1 \\ M_T(-1) = -1 \end{cases}}$$



these are easy to compute. solve a system of equations to find that

$$T' := \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix}$$ does the job. (or can just check that this works. uniqueness does the job for us)

Want det $= 1$ ⇝ $$\boxed{T := \frac{1}{\sqrt{1-\lambda^2}} \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix}} \in SL(2,\mathbb{R}) \subset SL(2,\mathbb{C})$$

Note that M_T restricts to a bi-holom. of $\mathbb{D}$ onto itself!

(the unit circle is the only circle passing through 1 that is perpendicular to the real axis.)

these are all preserved by M_T

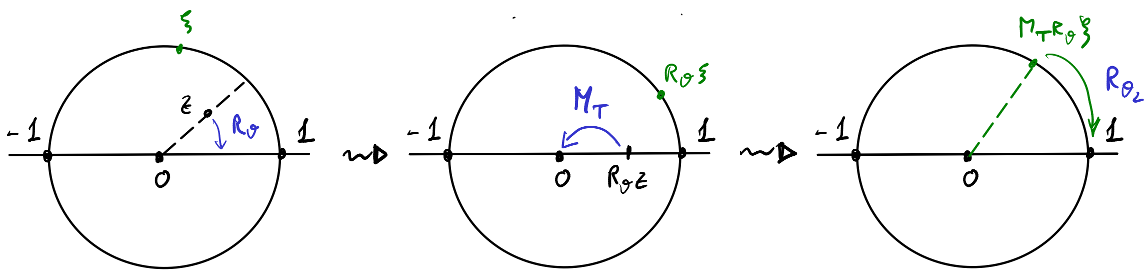
Let $G = \mathrm{Aut}(\mathbb{S}^2)$ and let $\mathrm{Stab}_G(\mathbb{D})$ be the group of Möbius transformations that restrict to biholomorphisms of $\mathbb{D}$ onto itself. We have:

Cor $\mathrm{Stab}_G(\mathbb{D})$ acts transitively on $\mathbb{D}$

Pf: given any $z \in \mathbb{D}$, it can be taken to the real axis via M_{R_θ} where $R_\theta = \begin{pmatrix} e^{i\theta'} & 0 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{C})$ gives a rotation of angle θ.

We can then use M_T as above to take $M_{R_\theta}(z)$ to 0. □

Note also that by postcomposing with a rotation, we can show that for every $\xi \in \partial\mathbb{D}$ there is a $F \in \mathrm{Stab}_G(\mathbb{D})$ s.t $F(z) = 0$ & $F(\xi) = 1$



Thm $\boxed{Aut(\mathbb{D}) \cong Stab_G(\mathbb{D})}$. That is, every biholomorphism of $\mathbb{D}$ is the restriction of a Möbius transformation.

Pf We will need the following: Ref: [Thm 12.2, Rudin]

Fact (Schwarz Lemma) suppose $f: \mathbb{D} \to \mathbb{D}$ is holomorphic s.t. $f(0)=0$.
Then $\boxed{|f(z)| \leq |z|}$ for every $z \in \mathbb{D}$, and $|f'(0)| \leq 1$

If we have equality for some $z \neq 0$ or $|f'(0)| = 1$, then
$\boxed{f(z) = \lambda \cdot z}$ for some $|\lambda| = 1$
(i.e. f is a rotation of the disk)

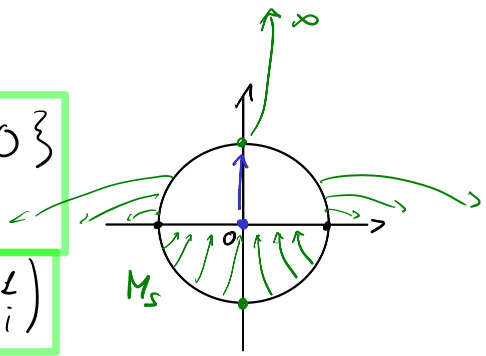
Now, pick any $F \in Aut(\mathbb{D})$. Postcomposing with a $M_A \in Stab_G(\mathbb{D})$, we can assume that $F(0)=0$.
We can hence use Schwarz Lemma on both F and F^{-1}.
Since $F'(0) \cdot (F^{-1})'(0) = Id'(0) = 1$, at least one of $|F'(0)|$ & $|(F^{-1})'(0)|$ is ≥ 1. By Schwartz lemma it must be 1, and hence F must be a rotation, which is in $Stab_G(\mathbb{D})$

$\square$

Rmk note that complex analysis it at play. A priori there's no reason why conformal maps of the disk should extend to conformal maps of S^2.

Note the Möbius Transformation $\boxed{\begin{array}{l} \mathbb{D} \xrightarrow{M_S} \{z \mid im(z) > 0\} \\ z \longmapsto \dfrac{iz-1}{-z+i} \end{array}}$ $\boxed{S = \begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix}}$

gives a conformal equivalence between $\mathbb{D}$ and the Upper Half Plane

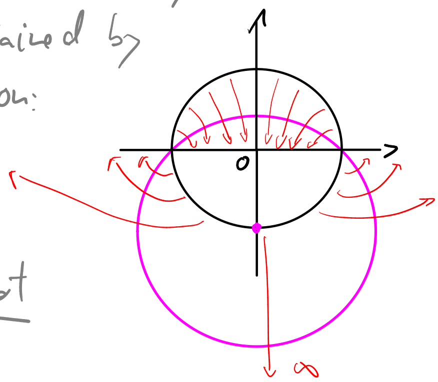


Exercise Show that $Stab_G(\text{Half Plane}) = PSL(2,\mathbb{R}) < PSL(2,\mathbb{C})$

$\boxed{\text{It follows that } \mathrm{Aut}(\mathbb{D}^2) \text{ is isomorphic to } \mathrm{PSL}(2,\mathbb{R}). \text{ In fact, } \mathrm{Aut}(\mathbb{D}^2) < \mathrm{PSL}(2,\mathbb{C}) \text{ is the image of } \mathrm{PSL}(2,\mathbb{R}) \text{ under conjugation by } M_S}$

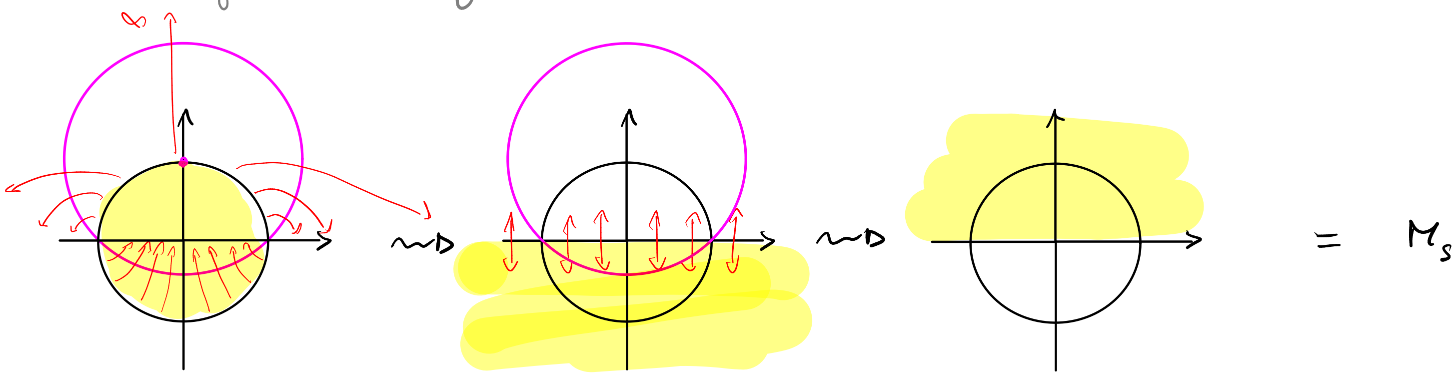
Recall It was an exercise in [Chap1 §3] to show that $\mathbb{D}$ with the Poincaré metric is isom to the upper half-plane with the Riemannian metric $\langle\cdot,\cdot\rangle = \frac{1}{y^2}\langle\cdot,\cdot\rangle_{Eucl}$ via the map $\mathbb{D}^2 \longrightarrow \{z \mid \mathrm{im}(z) > 0\}$ that is obtained by reflection:

$$(x,y) \longmapsto \left(\frac{2x}{x^2+(y+1)^2}, \frac{2(y+1)}{x^2+(y+1)^2} - 1\right)$$



This is not the same as M_S. In fact, it does not even preserve the orientation (and therefore it is not conformal according to our definition)

M_S can be described by composing a different sphere reflection with a reflection along the real axis:



§4 Classification of Möbius Maps

By §1 we know that every Riemann surface is conformally equivalent to X/Γ where $\Gamma < \mathrm{Aut}(X)$ is s.t. $\Gamma \curvearrowright X$ freely, properly discontinuously.
We know that $X = \mathbb{S}^2$, $\mathbb{C}$ or $\mathbb{D}$
and that $\mathrm{Aut}(X) = \mathrm{PSL}(2,\mathbb{C})$, $\underbrace{\{z \mapsto az+b\}}_{\substack{\wr \\ \mathbb{P}_{\mathbb{C}}(\text{Upper Triangular})}}$, $\mathrm{PSL}(2,\mathbb{R})$

It remains to understand what can $\Gamma < \mathrm{Aut}(X)$ be. We know that $\mathrm{Aut}(X)$ is always a group of Möbius transformations. It is therefore worthwhile studying such transformations more in detail.

Let $\boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in \underbrace{SL(2,\mathbb{C})}_{(ad-bc=1)}$ and $\boxed{M_A(z) := \dfrac{az+b}{cz+d}}$

Want to know: does M_A have fixed points? How many?

$z \in \mathbb{C} \cup \{\infty\}$ is a fixed point for M_A iff $z = \dfrac{az+b}{cz+d}$

i.e. iff $\boxed{cz^2 + (d-a)z + b = 0}$ ⇝ we can solve this!

The discriminant is: $(d-a)^2 - 4bc$

$= d^2 + a^2 + 2ad - 4(\underbrace{ad+bc}_{\det = 1}) = (a+d)^2 - 4$

$= \boxed{\mathrm{trace}(A)^2 - 4}$

(unless $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

$\Rightarrow$ $\boxed{\exists!\ \text{fixed point if } \mathrm{trace}(A)^2 = 4}$ ⇝ In this case, we say that M_A is **parabolic**

$\boxed{\exists\, 2 \text{ distinct fixed points if } \mathrm{trace}(A)^2 \neq 4}$

the fixed points may be ∞
(∞ is a fixed point $\Longleftrightarrow c = 0$)

conjugation preserves the trace

Let say M_A is parabolic. Up to conjugation, we can assume that the unique fixed point is ∞ (i.e. $c=0$). In this case M_A restricts to an autonorphism of $\mathbb{C}$ with no fixed points

$\Rightarrow$ $\boxed{\text{it must be a translation!}}$

Assume now that M_A is not parabolic. Up to conjugation, we can assume that the two fixed points are 0 and ∞. It follows that $\boxed{b=c=0}$ and $\boxed{d=a^{-1}}$ $\quad A=\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$

$\rightsquigarrow$ as an autonorphism of $\mathbb{C}$, M_A is given by $z \mapsto \lambda z$ for $\boxed{\lambda = a^2 \in \mathbb{C}\setminus\{0\}}$

There are a few qualitatively different cases:

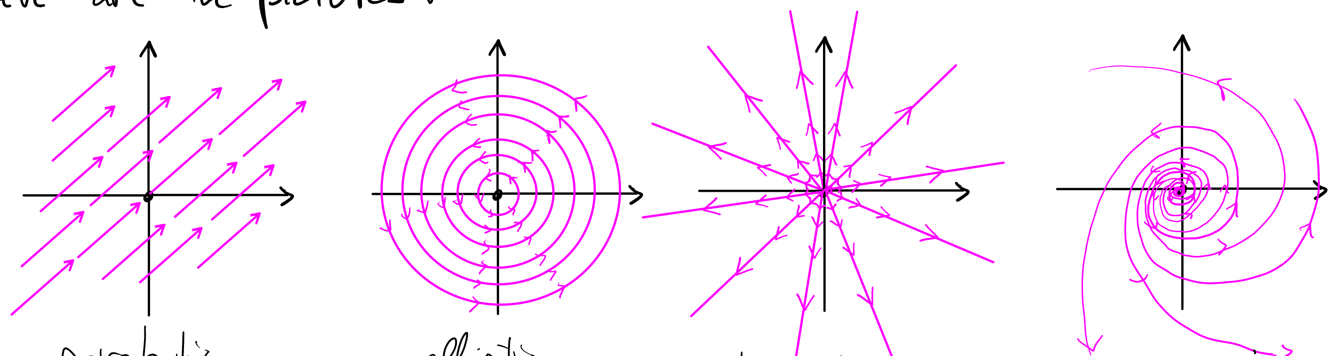
- Elliptic: $|\lambda| = 1$ (this is a rotation)
- Loxodromic:
  - $\boxed{\text{hyperbolic}}$ if λ is real, $0<\lambda<+\infty$ and $\lambda \neq 1$
  - $\boxed{\text{generic loxodromic}}$ if $\lambda \notin [0,\infty)$

Exercise Let $t_A = \mathrm{tr}(A)$ be the trace.

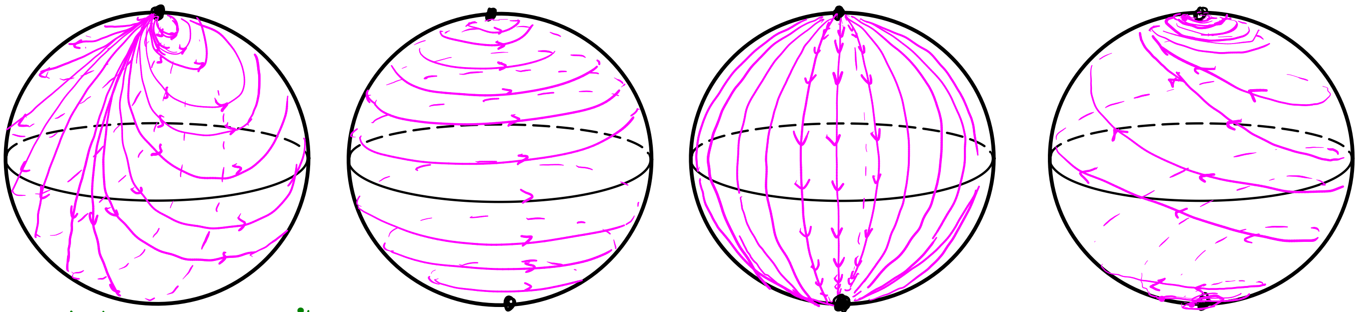
We know that M_A is parabolic iff $\boxed{t_A^2 = 4}$.

Show that
- M_A is elliptic $\Longleftrightarrow$ $\boxed{t_A^2 \in [0,4)}$
- " " hyperbolic $\Longleftrightarrow$ $\boxed{t_A^2 \in (4,\infty)}$ (i.e. is real and >4)
- " " general loxodromic $\Longleftrightarrow$ $\boxed{t_A^2 \in \mathbb{C}\setminus[0,\infty)}$

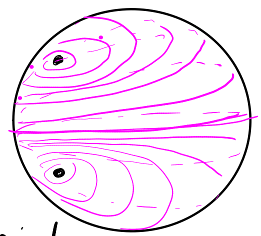
Here are the pictures:



parabolic $\mathrm{tr} = \pm 2$ | elliptic $\mathrm{tr} \in (-2,2$ | hyperbolic $\mathrm{tr} \in (-\infty,2) \cup (2,+\infty)$ | generic loxodromic $\mathrm{tr} \notin \mathbb{R}$



The orbits lie in a family of circles all tangent at the fixed point | The orbits lie in a family of parallel circles | The orbits lie in families of semi-circles converging at the fixed pts. | the orbits lie on families os spirals converging at the fixed points



Warning: the pictures above are very nice because we put the fixed points at the poles. If they are not on opposite sides the picture is a bit more complicated (but the topological-dynamical behaviour stays the same, of course)

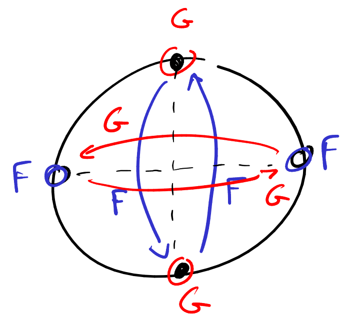
Since every $F \in PSL(2,\mathbb{C})$ has fixed points in S^2, we deduce:

Cor every Riemann surface whose universal cover is S^2, must be conformally equivalent to S^2 itself.

Rmk It actually follows by Brower fixed point Theorem that the sphere does not cover non-trivially any other surface.

For later use, it is convenient to know when Möbius Transformations commute. Let $\boxed{\mathrm{Fix}(F) := \{z \mid F(z) = z\}}$ for $F \in \mathrm{Aut}(\mathbb{S}^2)$

Lemma $F, G \in \mathrm{Aut}(\mathbb{S}^2)$ commute if and only if
either $\mathrm{Fix}(F) = \mathrm{Fix}(G)$
or F, G are elliptic of order 2 that "swap each other's fixed points"



proof since F, G commute, $\boxed{F(\mathrm{Fix}(G)) = \mathrm{Fix}(G)}$ ← $\mathrm{Fix}(G)$ is F-invariant (and vice versa)
and $\boxed{G(\mathrm{Fix}(F)) = G(\mathrm{Fix}(F))}$.

By staring at the picture, the following is clear:

Claim if F is parabolic or loxodromic then the only finite F-invariant subsets are the subsets of its Fixed-point set

- If both F & G are not elliptic then $\mathrm{Fix}(F) \subseteq \mathrm{Fix}(G)$ $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(F)$
  $\Rightarrow$ the fixed sets coincide
- If F is not elliptic and G is elliptic, then $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(F)$.
  (in particular F is not parabolic) $|\mathrm{Fix}(F)| = 2 \Rightarrow$ they must be equal.
- Let F and G be both elliptic. $F(\mathrm{Fix}(G)) = \mathrm{Fix}(G)$ can happen only if $\mathrm{Fix}(F) = \mathrm{Fix}(G)$ $\boxed{\text{or}}$ if F "swaps" the two points in $\mathrm{Fix}(G)$

In the latter case, $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) = \emptyset \Rightarrow G$ must "swap" $\mathrm{Fix}(F)$ as well. □

Exercise give a formal proof of the lemma using $\mathrm{Aut}(\mathbb{S}^2) \cong \mathbb{P}SL(2, \mathbb{C})$

Cor Parabolic elements can only commute with other parabolics

§5 Classification of Riemann surfaces $\mathbb{S}^2$ & $\mathbb{C}$

Recall: every Riemann surface Σ can be expressed as $X/_\Gamma$ where $X = \mathbb{S}^2, \mathbb{C}, \mathbb{D}$ and $\Gamma < \mathrm{Aut}(X)$ is a group of bi-holom. acting freely and prop. discontinuosly

($\Gamma \curvearrowright X$ is prop disc if $\forall x \in X$ $\exists$ a neighbourhood $x \in U$ s.t. $\{g \in \Gamma \mid g(U) \cap U \neq \emptyset\}$ is finite

We already noted that no element in $\mathrm{Aut}(\mathbb{S}^2)$ acts freely on $\mathbb{S}^2$
$\Rightarrow$ if $\Sigma = \mathbb{S}^2/_\Gamma$ then $\Sigma = \mathbb{S}^2$

Remains to understand how $X/_\Gamma$ looks like for $X = \mathbb{C}$ or $\mathbb{D}$

Let's study the case $\boxed{\mathbb{C} = X}$:

Note: $g \in \mathrm{Aut}(\mathbb{C})$ with no fixed points $\Rightarrow g$ must be a translation

$\Rightarrow$ $\Gamma < \mathrm{Aut}(\mathbb{C})$ acts freely if and only if it is a group of translations

Lem a group of translations $\Gamma < \mathrm{Aut}(\mathbb{C})$ that acts prop. disc. on $\mathbb{C}$ must be isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$

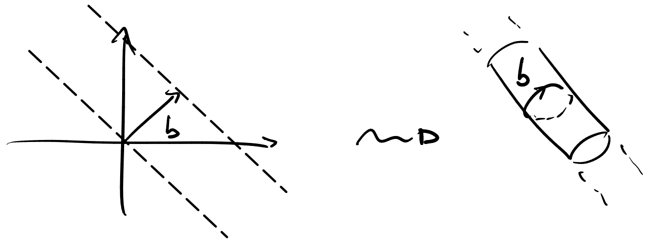
the proof is an **Exercise** (**Hint**: consider the elements $g \in \Gamma$ that displace the origin the least.)

Rmk that is, a subgroup of translations $\Gamma < \mathrm{Aut}(\mathbb{C})$ acts prop. disc. iff it is a discrete subgroup of $\mathbb{C} < \mathrm{Aut}(\mathbb{C}) < \mathrm{PSL}(2, \mathbb{C})$

it has no accumulation points

Case $\Gamma \cong \mathbb{Z}$:

Then Γ is generated by a translation $z \mapsto z+b$ $b \in \mathbb{C}$.
We can then identify $\mathbb{C}/\mathbb{Z}$ with an infinite cylinder of circumference b



We can also give another description of the Riemann surface $\mathbb{C}/\Gamma$ as follows: note that the map $\boxed{\mathbb{C} \longrightarrow \mathbb{C}^* = \mathbb{C}\setminus\{0\}}$ given by $\boxed{z \longmapsto \exp\left(\frac{2\pi i}{b} z\right)}$ is a holomorphic covering such that $\mathrm{Aut}(\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*)$ is precisely Γ

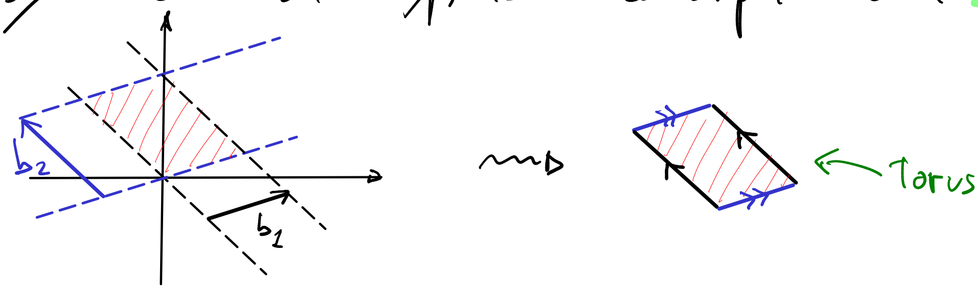
$\Rightarrow$ $\mathbb{C}^*$ is also a valid model for the Riemann surface $\mathbb{C}/\Gamma$

Rmk basically, we just remarked that the infinite cylinder of circumference b is conformally equivalent to $\mathbb{C}^*$. Since this does not depend on b, it also follows that all these cylinders are are conformally equivalent to one-another

Case $\Gamma \cong \mathbb{Z}^2$

Then Γ is generated by two translations $\boxed{\begin{array}{l} z \longmapsto z+b_1 \\ z \longmapsto z+b_2 \end{array}}$

It is quite easy to see that $\mathbb{C}/\Gamma$ is homeomorphic to a torus:



We thus proved the following:

Thm A Riemann surface Σ s.t $\tilde{\Sigma} \underset{\text{conf.}}{\cong} \mathbb{C}$ must be conformally equivalent to $\mathbb{C}$, $\mathbb{C}^*$ or a torus

these are all distinct, as they are not homeomorphic.

§6: Quotients of $\mathbb{D}$

It remains to study $\Gamma < \mathrm{Aut}(\mathbb{D})$. We begin by studying more carefully which Möbius transformations restrict to homeos of $\mathbb{D}$.

Lemma If $F \in \mathrm{Aut}(\mathbb{S}^2)$ is loxodromic, then it restricts to an automorphism of $\boxed{\mathbb{D}}$ $\Longleftrightarrow$ F is hyperbolic, with $\mathrm{Fix}(F) \subset \partial\mathbb{D}$

Pf: If F is loxodromic, note that for every $p \in \mathbb{S}^2$ we have

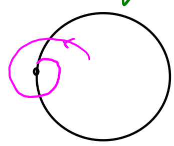
- $F^n(p) \xrightarrow{n\to\infty}$ one fixed point of F
- $F^{-n}(p) \xrightarrow{n\to\infty}$ the other fixed point

- Choosing p in $\mathbb{D}$ we deduce that $\mathrm{Fix}(F) \subseteq \overline{\mathbb{D}}$.
- Choosing $p \notin \mathbb{D}$ we deduce that $\mathrm{Fix}(F) \subseteq \overline{\mathbb{S}^2 \setminus \mathbb{D}}$

$$\boxed{\Longrightarrow \ \mathrm{Fix}(F) \subseteq \partial\mathbb{D}.}$$

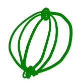
This picture does not mean that if F is a generic loxodromic then $\forall p\ \exists n$ s.t. $F^n(p)$ exits the circle. It has to be read as "there exists $p \in \mathbb{D}$ s.t. $F(p) \notin \mathbb{D}$"

Looking at the picture, F cannot be a generic loxodromic because the orbits "spiral around the fixed points"
$\rightsquigarrow$ F can't send $\mathbb{D}$ onto itself.



this is heuristic. The easiest way to make it into a formal argument is to conjugate so that $\mathrm{Fix}(F) = \{0, +\infty\}$ and show that no generalized lox. fixes the upper half plane

On the other hand, if F is hyperbolic, then it fixes every circle C s.t. $\mathrm{Fix}(F) \subset C$

(we noted that a hyperbolic el. fixes the meridians , and meridians are defined as circles in $\mathbb{S}^2$ passing through the poles (i.e. fixed pts))

$$\boxed{\text{It follows that if } F \text{ is hyperbolic with } \mathrm{Fix}(F) \subset \partial\mathbb{D} \text{ then it restricts to an autmorph. of } \mathbb{D}.}$$

$\square$

Exercise give a formal proof of the Lemma using $Aut(\mathbb{S}^2) \cong PSL(2,\mathbb{C})$
(*Hint*: it might be convenient to conjugate $\mathbb{D}$ onto the upper half-plane, so that $Aut(\mathbb{D})$ becomes $PSL(2,\mathbb{R})$)

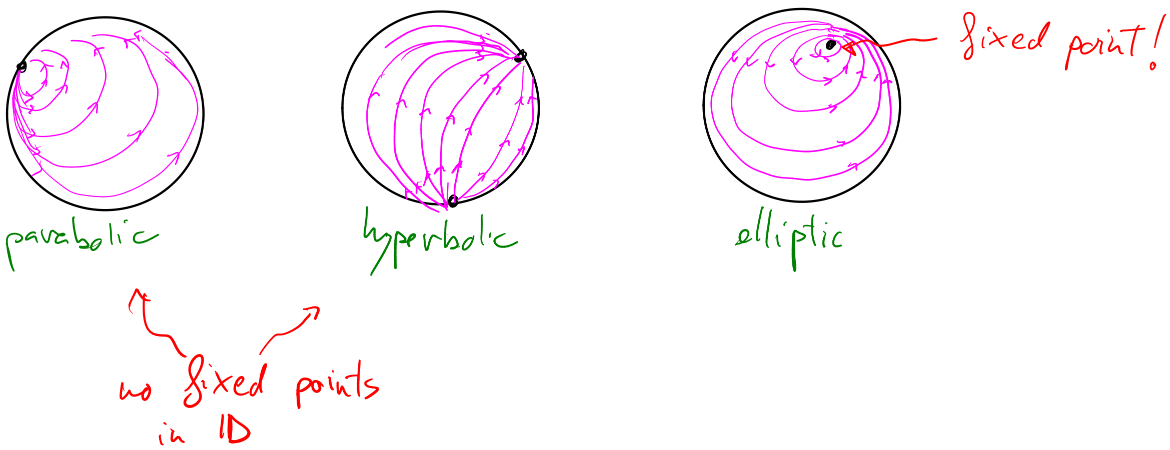
Rmk $\boxed{1)}$ if F is a parabolic element in $Aut(\mathbb{S}^2)$ and p is its fixed point, then there is a direction $v \in T_p\mathbb{S}^2$ such that F preserves all the circles that are tangent to p with direction v (and it preserves no other circles)

> In particular, F restricts to $Aut(\mathbb{D})$ iff its fixed point belongs to $\partial\mathbb{D}$ and $\partial\mathbb{D}$ is preserved under F.

$\boxed{2)}$ If $F \in Aut(\mathbb{S}^2)$ is an elliptic that restricts to $Aut(\mathbb{D})$, then it must have a fixed point in $\mathbb{D}$. (More precisely, F is a rotation around two "poles" that are symmetric w.r.t. $\partial\mathbb{D}$)

If you don't believe the remark, then it's an **Exercise**)

Bottom line: all the elements of $Aut(\mathbb{D})$ look like one of these:



Cor $\Gamma < \mathrm{Aut}(\mathbb{D})$ acts freely on $\mathbb{D}$ if and only if all its elements are parabolic or hyperbolic

Cor For every (topological) surface Σ, $\pi_1(\Sigma)$ is torsion-free

Pf: every surface Σ can be made into a Riemann surface $\rightsquigarrow$ Σ is homeomorphic to X/Γ $\Gamma < \mathrm{Aut}(X)$ $\Gamma \cong \pi_1(\Sigma)$ and $\Gamma \circlearrowright X$ free, prop. disc.

We just showed that $\Gamma < \mathrm{Aut}(X)$ cannot have elliptic elements for $X = \mathbb{S}^2, \mathbb{C}, \mathbb{D}$. All the other elements in $\mathrm{Aut}(X)$ have infinite order $\square$

This argument is an overkill, it is of course possible to prove this fact topologically. Note also that it is important that our definition of "surface" requires orientability. Non-orientable manifolds always have an element of order 2 in their fund. group.

To complete the classification of Riemann surfaces, one needs to classify subgroups of $\mathbb{P}SL(2,\mathbb{R})$ that are composed uniquely of parabolic and hyperbolic elements that acts prop. disc. on $\mathbb{D}$. This is very hard

Rmk one can prove (Exercise) that $\Gamma < \mathrm{Aut}(\mathbb{D}) < \mathbb{P}SL(2,\mathbb{C})$ acts prop. disc. iff it is a discrete subgroup. (Remember that the same was true for $\mathbb{C}$)... On the other hand, no infinite sgp of $\mathbb{P}SL(2,\mathbb{C})$ acts prop. disc. on $\mathbb{S}^2$

But we can classify the Riemann surfaces with abelian fundamental group!

Lemma An abelian subgroup of $\mathrm{Aut}(\mathbb{D})$ acting freely and prop. disc on $\mathbb{D}$ must be isom. to $\boxed{\mathbb{Z}}$

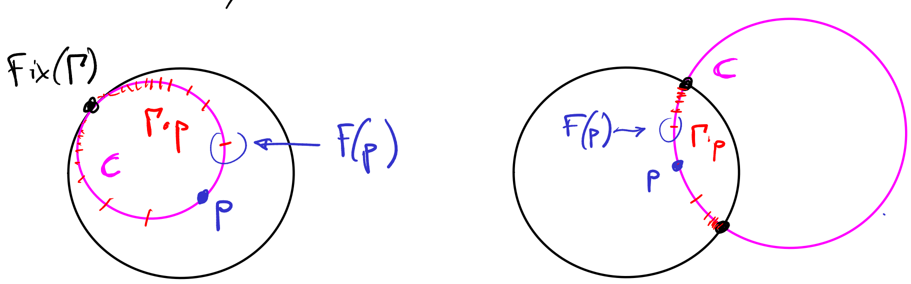
pf: Let $\Gamma < \mathrm{Aut}(\mathbb{D})$ be abelian. We know by §4 that $\boxed{\mathrm{Fix}(F) = \mathrm{Fix}(G)} = \mathrm{Fix}(\Gamma)$ $\forall F, G \in \Gamma$.

In particular, either Γ is composed only by parabolic elements
or it " " " " hyperbolic elements.

Choose any circle $\boxed{C \subset S^2}$ containing $\mathrm{Fix}(F)$, and a $\boxed{p \in C \setminus \mathrm{Fix}(F)}$.

It follows from our study of Möbius transformations, that $F(C) = C$ $\forall F \in \Gamma$ and that if $F(p) = p$ then $F = id$.

Since Γ acts prop. disc. (and $\mathbb{D}$ is loc. cpt.) there exists an $F \in \Gamma$ s.t. $F(p)$ is a "closest" point to p among all the points in $\Gamma \cdot p$
It is then easy to conclude that $\Gamma = \langle F \rangle$



□

Thm Let Σ be any Riemann surface. If $\pi(\Sigma)$ is abelian, then Σ is conformally equivalent to one of

$(\tilde{\Sigma} \cong S^2)$ $\boxed{\bullet}$ S^2

$(\tilde{\Sigma} \cong \mathbb{C})$ $\boxed{\bullet}$ $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$, a torus

$(\tilde{\Sigma} \cong \mathbb{D})$ $\boxed{\bullet}$ $\mathbb{D}$, $\mathbb{D}\setminus\{0\}$, the anulus $D_r = \{z \mid r<|z|<1\}$ for some $0<r<1$

If $\pi_1(\Sigma)$ is non-abelian then $\tilde{\Sigma}$ is conf eq to $\mathbb{D}$.

Pf: We know that $\tilde{\Sigma}$ is either S^2, $\mathbb{C}$ or $\mathbb{D}$. We alredy dealt with the first two cases. It thus remains to see what happens for $\mathbb{D}$.

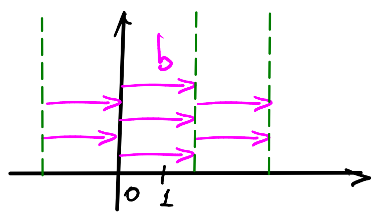
If $\pi_1(\Sigma)$ is abelian, then it must be generated by some $F \in \mathrm{Aut}(\mathbb{D})$.

It will be convenient to take conjugation so that $\mathbb{D}$ is sent to the upper half-plane.

If F is parabolic, we conjugate so that $\mathrm{Fix}(F) = \infty$

(formally, we conjugate Γ by a Möbius Transf that sends $\mathrm{Fix}(F)$ to ∞ and other 2 arbitrarily chosen pts of $\partial\mathbb{D}$ to 0 and 1)

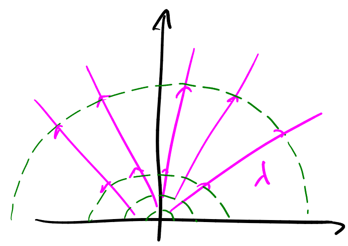
Then F takes the form $\boxed{z \longmapsto z+b}$ for some $b \in \mathbb{R}$.



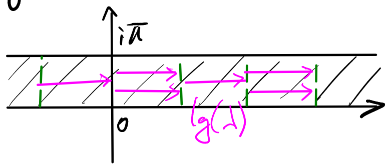
The map $z \longmapsto \exp\left(\frac{2\pi i}{|b|} z\right)$ descends to a conformal equivalence $\boxed{\text{UpperHalfplane}/\langle F\rangle \xrightarrow{\cong} \mathbb{D}\setminus\{0\}}$

Similarly, if F is hyperbolic we can assume that $\mathrm{Fix}(F) = \{0,\infty\}$ and hence F takes the form $\boxed{z \longmapsto \lambda z}$ for some $\lambda > 1$

we can assume this by swapping 0 and ∞ if necessary



We can choose a branch of the logarithm that makes the upper halfplane conf. eq. to a strip. Conjugating by this bi-holom. we see that the action becomes translation by $\log(\lambda)$.

$$\log(\text{Upper Half-plane}) =$$ 

Therefore, the funct. $z \longmapsto \exp\left(\frac{2\pi i}{\log(\lambda)} \log(z)\right)$ descends to a conf. eq.

$$\boxed{\mathbb{D}/\langle F\rangle \xrightarrow{\cong} D_r} \quad \text{for} \quad r = \exp\left(-\frac{2\pi^2}{\log(\lambda)}\right)$$

□

The following is an immediate consequence:

Cor If Σ is a (topological) surface with $\pi_1(\Sigma) \cong \mathbb{Z}$ then it is homeomorphic to a cylinder.
If $\pi_1(\Sigma) \cong \mathbb{Z}^2$ then Σ is homeomorphic to a torus

(Using this theorem to prove the corollary is clearly an overkill, but it is quite convenient)

§ 7: A naïve space of moduli

For us, the space of moduli on a surface will be the set of complex structures that can be put onto it up to conformal equivalence

$$\boxed{\mathrm{Moduli}(\Sigma) := \left\{ \begin{matrix} \text{Riemann surfaces} \\ \text{homeomorphic to } \Sigma \end{matrix} \right\} \Big/ \underbrace{\sim}_{\text{conf. eq}}}$$

Warning: In modern mathematics the term "moduli space" is used to denote a more complicated way of parametrizing structures on things, involving algebraic stacks, schemes and so on... We will not be concerned with this.

Example: If Σ is simply connected then it follows from the uniformization theorem that it is homeomorphic to the sphere or the disk. The moduli spaces are:

two cases

Σ is homeo to a sphere $\Longrightarrow$ $\boxed{\mathrm{Moduli}(\Sigma) = \{ S^2 \}}$

Σ is homeo to a disk $\Longrightarrow$ $\boxed{\mathrm{Moduli}(\Sigma) = \{ \mathbb{D}, \mathbb{C} \}}$

both of them are homeo to a disc, and are not conformally equivalent

We now wish to study Moduli spaces of surfaces that are not simply connected. The easiest way to do so, is by restating the problem using holomorphic coverings and groups of deck transformations.

Say that we are given two Riemann surfaces X/Γ_1, X/Γ_2 which we know are homeomorphic, and we are wondering whether they are conformally equivalent.

Claim 1) a conformal equivalence $F: X/\Gamma_1 \xrightarrow{\cong} X/\Gamma_2$ must lift to the universal cover. I.e. there exist a conformal equivalence $\tilde{F}: X \to X$ such that the diagram commutes:

$$\boxed{\begin{array}{ccc} X & \xrightarrow{\tilde{F}} & X \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X/\Gamma_1 & \xrightarrow{F} & X/\Gamma_2 \end{array}}$$

$F \circ \pi_1 = \pi_2 \circ \tilde{F}$

Rmk $\tilde{F}$ is not uniquely determined: to get a lift we need to choose some base points. ($\tilde{F}$ is only unique up to pre-composition by Γ_1 or post-composition by Γ_2)

Note that $\Gamma_1 < Aut(X)$ is the group of deck transformations $Aut(X \to X/\Gamma_1)$

self-conformal equivalences

They are more than "isomorphic": $\Gamma_1 = Aut(X \to X/\Gamma_1)$ as subsets of $Aut(X)$.

Claim 2) For every $\gamma \in \Gamma_1$, the biholomorphism $\tilde{F} \circ \gamma \circ \tilde{F}^{-1}$ belongs to Γ_2.

That is, conjugation by $\tilde{F}$ gives a homomorphism $\boxed{\Gamma_1 \to \Gamma_2}$.

Conjugation by $\tilde{F}^{-1}$ is an inverse $\Rightarrow$ This is an isomorphism.

Vice versa, if we are given $\tilde{F}: X \to X$ such that $\Gamma_2 = \Gamma_1^{\tilde{F}}$ then $\tilde{F}$ descends to a conformal equivalence $\boxed{F: X/\Gamma \to X/\Gamma_2}$

This denotes the conjugation: $\Gamma_1^{\tilde{F}} := \{\tilde{F} \circ \gamma \circ \tilde{F}^{-1} \mid \gamma \in \Gamma_1\}$

Putting together these observations, we obtain the following:

Lemma The moduli space of a (top.) surface Σ can be described as

$$\boxed{\left\{ \Gamma < \mathrm{Aut}(X) \;\middle|\; X = \mathbb{S}^2, \mathbb{C}, \mathbb{D}, \; X/\Gamma \underset{\text{homeo}}{\cong} \Sigma \right\} \Big/ \sim}$$

where $\Gamma_1 < \mathrm{Aut}(X_1) \sim \Gamma_2 < \mathrm{Aut}(X_2)$ iff. $\exists \tilde{F} : X_1 \xrightarrow{\cong} X_2$ s.t. $\Gamma_2 = \Gamma_1^{\tilde{F}}$

(In particular we have $\Gamma \cong \pi_1(\Sigma)$)

Exercise prove in detail the claims & the lemma.

Lemma $\mathrm{Moduli}(\text{cylinder}) = \{ \mathbb{C}^*, \mathbb{D} \setminus \{0\} \} \cup \{ D_r \mid 0 < r < 1 \}$

Pf: We know by the classification of Riemann surfaces with abelian fundamental group, that if X/Γ is a cylinder then X/Γ is conf. eq. to one of these (because $\Gamma \cong \pi_1(X/\Gamma) \cong \mathbb{Z}$).
All that remains to show is that these are all distinct points in the moduli space (i.e. they are not conformally equivalent to one another)

Since conformal equivalences lift to the universal cover, ($X_1/\Gamma_1 \cong X_2/\Gamma_2 \implies X_1 \cong X_2$) we deduce that $\mathbb{C}^*$ is different from $\mathbb{D} \setminus \{0\}$ and D_r.

Both $\mathbb{D} \setminus \{0\}$ and D_r are of the form $\mathbb{D}/\Gamma$ with $\Gamma \cong \mathbb{Z}$, but for $\mathbb{D} \setminus \{0\}$ Γ consists of parabolic elements, while for D_r it consists of hyperbolic $\implies$ they cannot be conjugated in $\mathrm{Aut}(X)$

$\implies \mathbb{D} \setminus \{0\} \neq D_r \quad \forall\, 0 < r < 1$

It remains to show that all the D_r are distinct.
i.e. $D_{r_1} = D_{r_2} \implies r_1 = r_2$

Remember that we showed explicitly that $\boxed{D_{r_1} \cong \text{Half Plane} / \langle z \mapsto \lambda_1 z \rangle}$
where $\boxed{\lambda_1 = \exp\left(-\frac{2\pi^2}{\log(r_1)}\right)}$, and similarly for D_{r_2}

That is, $D_{r_1} = \text{Half Plane} / \Gamma_1$ and $D_{r_2} = \text{Half Plane} / \Gamma_2$

where $\Gamma_1 = \langle \underbrace{z \mapsto \lambda_1 z}_{= a_1} \rangle$ $\quad \Gamma_2 = \langle \underbrace{z \mapsto \lambda_2 z}_{= a_2} \rangle$

If $D_{r_1} = D_{r_2}$ then Γ_1 and Γ_2 must be conjugate
i.e. $\exists \tilde{F}$ s.t. $\Gamma_2 = \Gamma_1^{\tilde{F}}$.
Since the only elements that generate $\mathbb{Z}$ are $+1$ and -1,
we deduce that the conjugate $a_1^{\tilde{F}}$ must be either a_2 or a_2^{-1}

On the other hand, recall that a_1 is given by the Möbius transformation $\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_1}^{-1} \end{pmatrix}$ whose trace is $\sqrt{\lambda_1} + \frac{1}{\sqrt{\lambda_1}}$

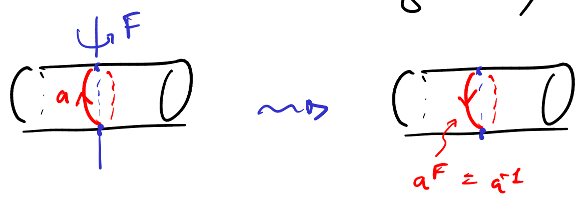
The trace is invariant under conjugation

$$\implies tr(a_1) = tr(a_1^{\tilde{F}}) = \begin{cases} tr(a_2) \\ tr(a_2^{-1}) = tr(a_2) \end{cases}$$

The unique solution $\lambda_2 > 1$ for the equation $\boxed{\sqrt{\lambda_2} + \frac{1}{\sqrt{\lambda_2}} = \sqrt{\lambda_1} + \frac{1}{\sqrt{\lambda_1}}}$
is $\lambda_1 = \lambda_2 \implies \boxed{r_1 = r_2}$

$\square$

Rmk it can indeed happen that $\tilde{F}(a_1) = a_2^{-1}$. Conjugating by an appropriate rotation by π we can send a_1 to a_1^{-1}.
This descends to a rotation of the cylinder:



We now want to study the moduli space of the torus .
Since $\boxed{\pi_1(\text{torus}) \cong \mathbb{Z}^2}$ is abelian, we can apply the classification of Riemann surfaces with abelian fundamental group and we deduce that $\mathbb{C}$ is the only simply connected Riem. surface X such that $X/\Gamma \underset{\text{homeo}}{\cong}$ torus for some $\Gamma < \mathrm{Aut}(\mathbb{C})$.
Further, we know that $\mathbb{C}/\Gamma \cong$ torus iff $\Gamma < \mathrm{Aut}(\mathbb{C})$ is a discrete group of translations and $\Gamma \cong \mathbb{Z}^2$

we need this so that $\Gamma \curvearrowright \mathbb{C}$ is prop. disc.

$$\Longrightarrow \boxed{\mathrm{Moduli}(\text{Torus}) = \{\Gamma < \mathrm{Aut}(\mathbb{C}) \mid \Gamma < \mathbb{C} \text{ discrete}, \ \Gamma \cong \mathbb{Z}^2\}/\sim}$$

Γ is a group of translations

How can we describe this set? Let's begin by better understanding $\sim$. That is, we want to understand what happens when we conjugate a translation $(z \mapsto z+a)$ by some $\tilde{F} \in \mathrm{Aut}(\mathbb{C})$.

- If $\tilde{F}$ is a translation $\tilde{F} = (z \mapsto z+b)$, then

$$\underbrace{(z \mapsto z+a)^{\tilde{F}}}_{\tilde{F} \circ (z \mapsto z+a) \circ \tilde{F}^{-1}} = (z \mapsto ((z-b)+a)+b) = (z \mapsto z+a)$$

That is, conjugating by a translation doesn't do anything on $(z \mapsto z+a)$

- If $\tilde{F}$ is a complex multiplication $\tilde{F} = (z \mapsto \alpha z)$ for some $\alpha \in \mathbb{C}^*$ (i.e. $\tilde{F}$ is a rotation + dilatation) then

$$(z \mapsto z+a)^{\tilde{F}} = (z \mapsto (\alpha(\alpha^{-1}z)+a)) = (z \mapsto z + \alpha a)$$

That is, conjugating by $(z \mapsto \alpha z)$ sends the translation $z \mapsto z+a$ to the translation by the complex number αa.

(the action by conjugation of $\mathbb{C}^* < \mathrm{Aut}(\mathbb{C})$ on $\mathbb{C} < \mathrm{Aut}(\mathbb{C})$ coincide with the action by complex multiplication $\mathbb{C}^* \curvearrowright \mathbb{C}$)

Since $Aut(\mathbb{C})$ is generated by roto-dilatations & translation, we deduce that $\tilde{F}$ conjugates $(z \mapsto z+a)$ to $(z \mapsto z+b)$ if and only if $\tilde{F}$ is a roto-translation by $\frac{b}{a}$ (up to translation)

More formally, what we showed is that the action by conjugation descends to an action of the quotient:

$$\begin{tikzcd} Aut(\mathbb{C}) \arrow[d] & \circlearrowright & \{translations\} = \mathbb{C} \\ \mathbb{C}^* = Aut(\mathbb{C})/\mathbb{C} \arrow[urr, dashed, "\circlearrowright"] & & \end{tikzcd}$$

A note about notation: when I write $\mathbb{C} < Aut(\mathbb{C})$ it means that I am identifying the subgroup of translations with $\mathbb{C}$. Similarly, the group of roto-dilatations that fix the origin is identified with $\mathbb{C}^*$.
The fact that $\mathbb{C}$ and $\mathbb{C}^*$ generate $Aut(\mathbb{C})$, $\mathbb{C} \triangleleft Aut(\mathbb{C})$ and $\mathbb{C} \cap \mathbb{C}^* = \{id\}$ imply that $Aut(\mathbb{C})/\mathbb{C} \cong \mathbb{C}^*$. In other words, $Aut(\mathbb{C})$ is isomorphic to the semi-direct product $\boxed{\mathbb{C} \rtimes \mathbb{C}^*}$

↑ you can take the above description as a definition.

Bottom line: if we identify {translations} with $\mathbb{C}$ and we care about things up to conjugation in $Aut(\mathbb{C})$, then this is the same as looking $\mathbb{C}$ up to multiplication by some (fixed) complex number.

For example, if we are given two ordered sets of translations

$$\gamma_1 = (z \mapsto z + a_1) \qquad \gamma_1' = (z \mapsto z + a_1')$$
$$\gamma_2 = (z \mapsto z + a_2) \qquad \gamma_2' = (z \mapsto z + a_2')$$
$$\vdots \qquad\qquad\qquad \vdots$$

and we wonder whether there exists $\tilde{F} \in Aut(\mathbb{C})$ such that $\gamma_i^{\tilde{F}} = \gamma_i' \ \forall i$, this is very easy to check: the only (up to translation) $\tilde{F}$ s.t. $\gamma_1^{\tilde{F}} = \gamma_1'$ is multiplication by $\frac{a_1'}{a_1}$.

That $\tilde{F}$ sends γ_i to γ_i' $\forall i$ iff $\frac{a_i'}{a_i} = \frac{a_1'}{a_1}$ $\forall i$

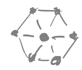
discrete

Back to our problem: we are given $\Gamma_1, \Gamma_2 < \mathbb{C} < \mathrm{Aut}(\mathbb{C})$, $\Gamma_1 \cong \mathbb{Z}^2 \cong \Gamma_2$ and we want to know whether there is $\tilde{F} \in \mathrm{Aut}(\mathbb{C})$ such that $\Gamma_2 = \Gamma_1^{\tilde{F}}$.

This is quite hard! the problem is that given some $\gamma \in \Gamma_1$ we don't know a priori what element of Γ_2 should be $\gamma^{\tilde{F}}$ and hence we can't immediately use the cheap argument of above.

Idea: how did we manage to solve the problem of moduli for the cylinder? We had chosen a generator $a_1 \in \Gamma_1$ and we had noted that if $\Gamma_2 = \Gamma_1^F$ then a_1^F must be a generator of Γ_2.
Γ_2 has only two generators $\Rightarrow a_1^F = a_2$ or a_2^{-1}.

Now, we can't do this in the torus case, because $\mathbb{Z}^2$ has infinitely many pairs of generators. The idea is then to look for some "special" set of generator that we can tell apart from all the other ones.

The key observation is that $\Gamma_1 < \mathbb{C}$ will have only finitely many elements of minimal norm. (at most 6 : )
Since the action by conjugation is a multiplication by one fixed complex number,

$$\boxed{\text{an element of minimal norm of } \Gamma_1 \text{ must be sent to an element of minimal norm of } \Gamma_2 \text{ !}}$$

Knowing this, describing the moduli space of the torus is not hard.

this will be important later.

Given $\Gamma_1 < \mathbb{C}$ discrete, $\Gamma_1 \cong \mathbb{Z}^2$ choose $a \in \Gamma_1$ of minimal complex norm.

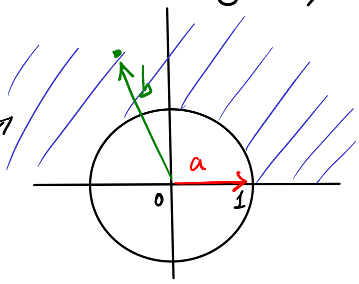
$$|a| = \min\{|\gamma| \mid \gamma \in \Gamma\}.$$

For convenience, note that $\exists! \, \tilde{F} \in \mathbb{C}^* < \mathrm{Aut}(\mathbb{C})$ such that $a^{\tilde{F}} = (z \mapsto z+1)$
$\Rightarrow$ conjugating by $\tilde{F}$ we can assume that $a = (z \mapsto z+1)$

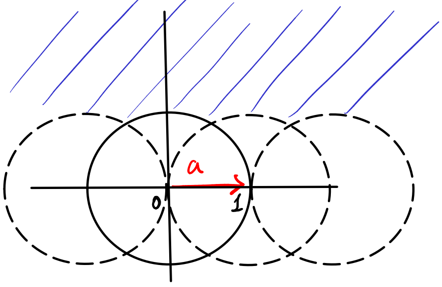
We now need to choose a second element $\boxed{b \in \Gamma < \mathbb{C}}$ such that $\Gamma = \langle a, b \rangle$.
We can choose it to be an element in $\Gamma \setminus \langle a \rangle$ of minimal complex norm and with positive imaginary part.

this condition is equivalent to saying that the angle (with sign) from a to b is in $(0, \pi)$
$$0 < \angle(a,b) < \pi$$

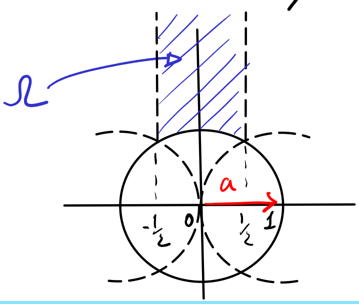
Since $|b| \geq |a|$, b must be somewhere here



Further, we know that be cannot be at distance < 1 from a^n for any $n \in \mathbb{Z}$, otherwise ba^{-n} would be an element of complex norm less than $|a|$.



Finally, b must belong to a thin vertical strip, otherwise we could get an element with smaller norm by composing with $a^{\pm 1}$.
Let $\boxed{\Omega}$ be that strip.



That is, we just showed that every point in the moduli space of the torus has a representative $\Gamma < \mathrm{Aut}(\mathbb{C})$ such that Γ is generated by $(z \mapsto z+1)$ and $(z \mapsto z+b)$ for some $b \in \Omega$.

It only remains to understand whether different b's correspond to the same point in the moduli space or not.

That is, we want to understand for what $b, b' \in \Omega$ we can have that $\Gamma_1 := \langle a, b_1 \rangle$ and $\Gamma_2 := \langle a, b_2 \rangle$ are conjugate.

Saying "every el. of Mod.Sp. has a representative of the form..." means that, as a set, $\mathrm{ModSp(torus)} = \Omega/\sim$ for some equivalence relation $\sim$.
We are now trying to understand $\sim$.

Note that since b is defined as "the smallest s.t. ..." if $\Gamma_2 = \Gamma_1^{\tilde F}$ then $\boxed{|b_1| = |b_2|}$

(we already normalized Γ_1 and Γ_2 s.t. $\min\{|\gamma| \mid \gamma\in\Gamma_1\} = |a| = \min\{\gamma \mid \gamma \in \Gamma_2\}$)

Case $|b_1| > |a|$. In this case, if $\Gamma_2 = \Gamma_1^{\tilde F}$ we must have $a^{\tilde F} = a$ or $-a$

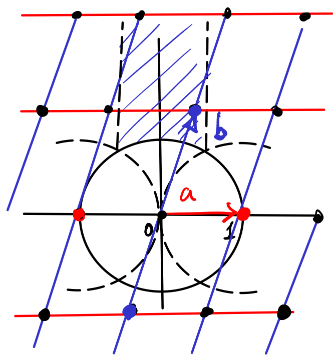
Note that the conjugation by $z \mapsto -z$ sends a to $-a$ and leaves Γ_1 invariant i.e. $\Gamma_2^{(z\mapsto -z)} = \Gamma_2$

$\Longrightarrow$ Pre-composing with this conjugation if necessary, we can assume that $a^{\tilde F} = a \Longrightarrow \tilde F$ is the identity.

Bottom line: if $|b_1| = |b_2| > |a|$, then $\langle a, b_1\rangle$ and $\langle a, b_2\rangle$ are conjugate iff they are equal (as subsets of $\mathbb{C}$)

$$\Gamma_1 = \Gamma_2 = \Gamma < \mathbb{C}$$

Subcase 1) b_1 is in the interior of Ω.

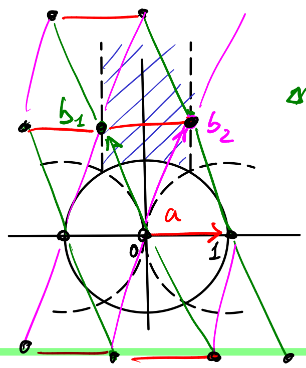


In this case it is easy to see that every other $b \in \Omega \cap \Gamma$ has $|b| > |b_1| \Longrightarrow$ since $|b_2| = |b_1|$ we must have $b_2 = b_1$

Bottom line: every point in the interior of Ω uniquely determines a point in the moduli space.

Subcase 2) $b_1 \in \partial\Omega$.

It is then easy to see that there are pairs of distinct points s.t. $b_1, b_2 \in \partial\Omega$ with $b_2 = b_1 + 1 = b_1 + a$ and hence generate the same group Γ. $\rightsquigarrow$ those two points determine the same element in the moduli Space (and different pairs determine different elements)

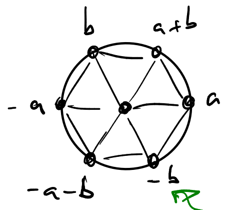


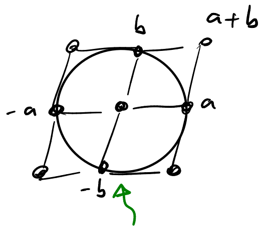
here the group consists of the black points.

The coloured lines represent our choice of generators

Case $|b_1| = |a|$ Now we cannot assume that $a^{\tilde F} = a$.

Luckily, there are not too many other possibilities because there can be at most 6 points of Γ_1 on the circle:



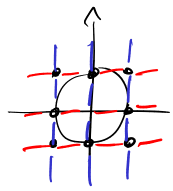


6 points. Special case   4 points: general case

Subcase 1: $b_1 = (z \mapsto z + i)$.

If $\Gamma_2 = \Gamma_1^{\tilde F}$ then $a^{\tilde F} \in \{a, -a, b, -b\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1 \quad -1 \quad i \quad -i$



Conjugating by $(z \mapsto -z)$ $(z \mapsto iz)$ $(z \mapsto -iz)$ we can send a to $-a$, b, $-b$.

These conjugations leave Γ_1 invariant $\Rightarrow$ up to precomposing we can assume that $\tilde F = id$ i.e. $\Gamma_1 = \Gamma_2$.

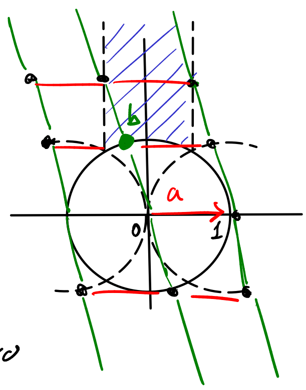
$\boxed{\text{It then follows as above that the point } b = i \text{ uniquely determines an element of the moduli space}}$

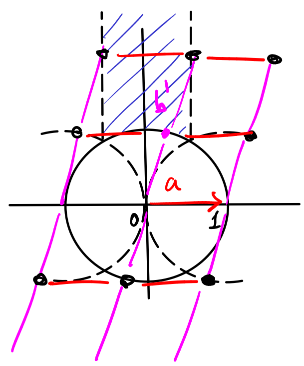
Subcase 2: $b = e^{i\vartheta}$ for $\vartheta \neq \pi/3, \pi/2, 2\pi/3$

Again we have that if $\Gamma_2 = \Gamma_1^{\tilde F}$ then $a^{\tilde F} \in \{a, -a, b, -b\}$

The conjugation taking a to $-a$ and b to $-b$ leave Γ_1 invariant $\Rightarrow$ wlog $a^{\tilde F} \in \{a, b\}$.

On the other hand, the conjugation taking b to a does not leave Γ_1 invariant $\rightsquigarrow$ we cannot reduce to the case where $\tilde F = id$.





WLOG $\Rightarrow$ $\tilde F$ is either (id) or $(z \mapsto bz)$

(this is the rotation sending a to b)

These 2 $\tilde F$ do indeed give us two different $\Gamma_1, \Gamma_2 < \mathbb{C}$ that are conjugate

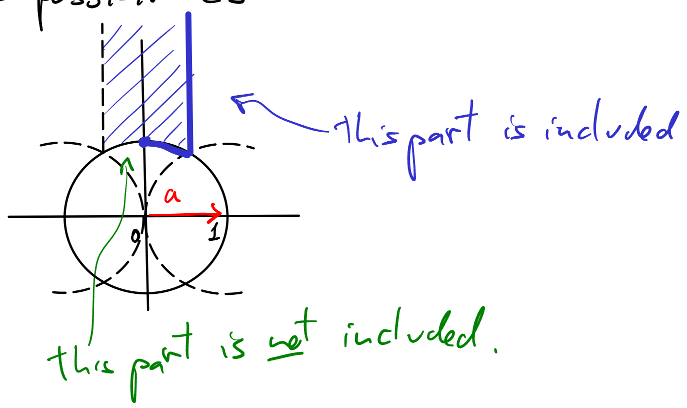
$\Rightarrow$ the points $b_1 = e^{i\vartheta}$ and $b_2 = e^{i(\frac{\pi}{2}-\vartheta)}$ determine the same point in the moduli space.

On the other hand two points $e^{i\vartheta_1}$, $e^{i\vartheta_2}$ with $\pi/3 < \vartheta_1 < \vartheta_2 < \pi/2$ will determine different elements of the moduli space.
In fact the set $\{\angle(\gamma,\gamma') \mid \gamma,\gamma' \in \Gamma_1 \text{ elements of minimal norm}\}$ is equal to $\{\vartheta_1, -\vartheta_1, \pi-\vartheta_1, -\pi-\vartheta_1\}$ and is invariant under conjugation.

Final Subcase: $b = e^{i\vartheta}$ $\vartheta = \pi/3$ or $2\pi/3$.

In this case we can again assume that $\tilde{F} = id$
$\rightsquigarrow$ these two points determine the same element in the moduli space by the same argument of Subcase 2 of case $|b| > |a|$.

This case-check exhausts all the possibilities
$\Rightarrow$ we can say that
Moduli Space (torus) can be identified (as a set) with this



Just seeing it as this set is somewhat unnatural though. it is much better to realize it as the quotient of Ω where we identify points that yield the same element in the moduli space.
What we get is a sphere with one cusp and 2 special points: