2: Structures on Surfaces

§1: Basic definitions

Def A *n-dimensional manifold with boundary* is a Hausdorff 2^{nd}-countable topological space M such that every point $p \in M$ has a neighbourhood U that is homeomorphic to $\mathbb{R}^n$ or the closed upper-half space

The *boundary* of M is the set of points that don't have a neighbourhood homeo. to $\mathbb{R}^n$. It's denoted by ∂M.

We also use the notation $(M, \partial M)$ for manifolds with boundary. Note that ∂M is a $n-1$ dimensional manifold.

[Hausdorff, 2^{nd} c'table + loc. homeo to $\mathbb{R}^n$] $\Longleftrightarrow$ [separable, metrizable + loc. homeo to $\mathbb{R}^n$]

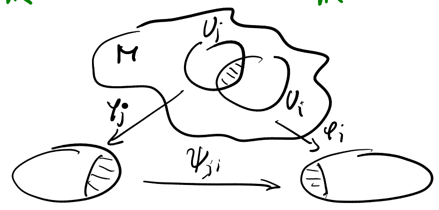
"$\Rightarrow$" 2^{nd} c'table + loc. homeo to $\mathbb{R}^n$ $\to$ paracompact. "$\Leftarrow$" obvious

Hausdorff + paracompact $\to$ metrizable

2^{nd} c'table $\to$ separable

A (smooth) *differentiable structure* on M is given by an atlas of *local coordinates* — a covering by connected, open sets U_i of M together with maps $\boxed{\varphi_i : U_i \to \mathbb{R}^n}$ or <upper half space> — such that the change-of-coordinates maps $\boxed{\psi_{ji} := \varphi_j(U_i \cap U_j) \xrightarrow[\varphi_i \circ \varphi_j^{-1}]{} \varphi_i(U_i \cap U_j)}$ are C^∞-diffeomorphisms (we will not be concerned with C^k and such...)

$\varphi_j(U_i \cap U_j) \subset \mathbb{R}^n$, $\varphi_i(U_i \cap U_j) \subset \mathbb{R}^n$



We will now restrict to 2-dimensional manifold. We use the following:

Def a *surface* $(\Sigma, \partial\Sigma)$ is a connected, orientable 2-dimensional manifold.

Rmk this choice is made out of convenience, because these are the objects we care about. If we'll ever need to consider non-orientable or disconnected surfaces I will say it explicitely

Defining property what it means for a manifold to be orientable is kind of tricky. If M is compact, this can be defined in term of existence of a fundamental class in top-dimensional homology.
If M is differentiable, it can be defined in terms of tangent bundles, or determinants of change-of-coordinate maps.
Since we are only dealing with surfaces, an easy way out is to say that a surface is $\boxed{\text{orientable} \iff \nexists \text{ homeomorphic embedding of the Möbius strip}}$

Def A surface with empty boundary is a Riemann surface if it has a complex structure.
That is, a differentiable structure such that the change-of-coordinate maps are bi-holomorphisms (i.e. ψ_{ij} is a conformal equivalence between $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$)

Rmk
- One could also define Riemann surfaces with boundary, but most people don't. And neither will we.
- The orientability assumption is redundant: if a 2-dimensional manifold has a complex structure then it is automatically orientable (holomorphic maps preserve the natural orientation of $\mathbb{C}$)
  y, $\hat{x}$ multiply by i
- Some authors (e.g. Ahlfors-Sario) don't assume that the manifold be a 2^{nd} ctable topological space, and then they prove that a Riemann surface must indeed be 2^{nd} ctable.

Finally, we say:

Def a Riemannian metric on a surface $(\Sigma, \partial\Sigma)$ is given by a choice of Riemannian metrics on the charts $\varphi_i(U_i) \subseteq \mathbb{R}^2$ such that the change-of-coordinate maps are Riemannian isometric.
A surface with a Riemannian metric has geodesic boundary if $\partial\Sigma$ consists of geodesics

Rmk • a Riemannian metric on a surface can be used to define the length of piece-wise smooth curves. We thus get an actual metric on Σ, just as in the case of Riem. metrics of planar domains

- Since we were quite sloppy with the definitions, some care is needed for the boundary points. If $p \in \partial\Sigma$ and $p \in U_i$ then the Riemannian metric should be defined in a neighbourhood of $\varphi_i(U_i)$, so that we have nice tangent space at $\varphi_i(p)$ and a norm $\|\cdot\|_{\varphi(p)}$ on it.

- We already remarked that Riemannian-isometric maps send geodesics to geodesics. In particular, it is well defined to say that a curve $\gamma: I \to \Sigma$ is a geodesic if its image in every coordinate chart is a geodesic. Note also that $\partial\Sigma$ is a collection of curves and it therefore make sense to ask whether they are geodesics.

Rmk given any $\Omega \subset \Sigma$ open, connected and $\psi: U \longrightarrow \Omega \subseteq \mathbb{R}^2$ diffeomorphism, there is a unique Riemannian metric on Ω such that the maps $\psi \circ \varphi_i^{-1}$ are all isometric embedding (where defined).

Rmk our convention is quite awkward, because we are describing Riemannian metrics only in terms of local coordinates. But you really need to think of the Riemannian metric to be defined on the surface itself. This is the meaning of the above remark.

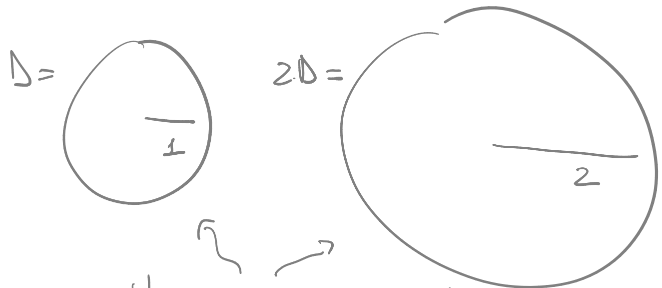
Once again: Riemann $\neq$ Riemannian

In fact, if we are given maps $\psi_{ij} : \underset{\mathbb{C}}{\underset{\cap}{\varphi_i(U_i \cap U_j)}} \longrightarrow \underset{\mathbb{C}}{\underset{\cap}{\varphi_j(U_i \cap U_j)}}$

that are biholomorphism, there is no reason to believe that we can find Riemannian metrics on $\varphi_i(U_i)$ such that ψ_{ij} are Riemannian isometries.

Conversely, a priori there is no reason to believe that if we are given Riemannian metrics such that ψ_{ij} are isometries then the ψ_{ij} are biholomorphisms
(if the metrics on $\varphi_i(U_i)$ are not conformally equivalent to the Euclidean metric the maps ψ_{ij} need not be holomorphic)

It will turn out that it is actually possible to show that any Riemannian structure induces a unique Riemann structure.
The converse is even more delicate:



with Euclidean metric

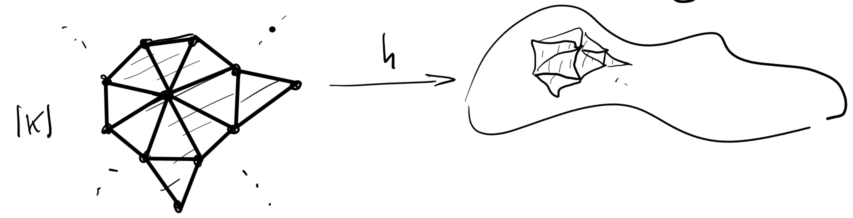
these are isomorphic (conformally equivalent) Riemann surfaces,
but they are not isomorphic (i.e. isometric) Riemannian surfaces.
How do you choose which one of them should be the 'correct one'?
(there's a way to get around this once the classification of Riemann surfaces is known.)

We will say more on this later on...

§2: Topology of surfaces

In order to classify surfaces, it is convenient to talk about triangulations.

Def A *triangulation* of a manifold $(M, \partial M)$ is a homeomorphism $h : |K| \longrightarrow M$, where K is a countable, locally finite simplicial complex and $|K|$ is its geometric realization.



The following is a tricky theorem. See [Moise, Chapter 8 Thm.5] for a proof.

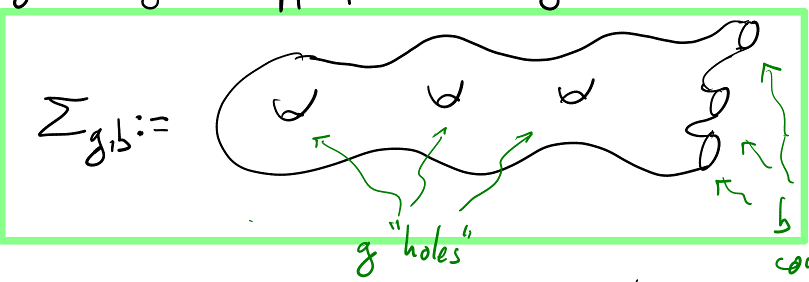
Thm (Rado) Every 2-dimensional manifold admits a triangulation.

The analogous theorem is valid also in dimension 3 and is due to Moise. It is false in higher dimension (E_8 does not have any triangulation. The double suspension of a Poincaré sphere has a triangulation that is not PL (see below))

Assuming the existence of triangulations, the classification of compact surfaces is fairly simple and it is left as a (guided) exercise.

Thm (Classification of compact surfaces) Every compact surface is homeomorphic to $\Sigma_{g,b}$ for appropriate $g, b \in \mathbb{N}$.

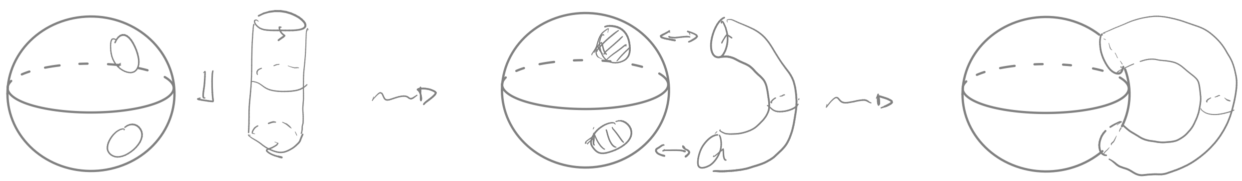
Where



is the surface of genus g with b boundary components.

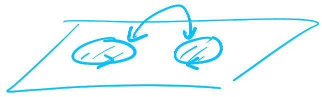
Defining the genus formally requires a bit of care (depending on how much topology you know, that is).

Genus is obtained by picking 2 disks in a surface, removing them and gluing back a cylinder instead. This procedure is called "adding a handle"



topologically, this process is equivalent to removing 2 disks and identifying the resulting boundary components

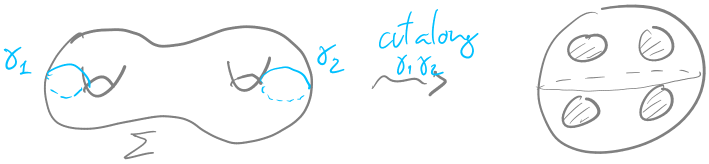
the gluing need to be compatible with the orientation



i.e. injective map $S^1 \to \Sigma$

The inverse process can be seen as "cutting" along a simple closed curve (and "filling in" the boundaries with disks)

> We can thus define the genus of a surface Σ as the maximal number of disjoint simple closed curves $\gamma_i : S^1 \to \Sigma$ such that $\Sigma \setminus \{\gamma_1, \dots, \gamma_n\}$ is still connected



there are no more simple closed curve that don't disconnect

Formally, $\Sigma_{g,b}$ is defined as the surface obtained from a sphere by adding g handles and removing b (open disks)

here we are implying that the choices (which disks to cut) we are making don't influence the outcome (up to homeomorphism)

To prove this formally one can use the Jordan-Schönflies theorem. Alternatively, one could define "attaching handles" for triangulated surfaces. Then it becomes more elementary to show that the outcome doesn't depend on the choices.

(i.e. removing triangles instead of disks)

Def the Euler Characteristic $\chi(\Sigma)$ of a surface Σ is the alternating sum of its Betti numbers (rank of homology groups). Concretely, if $\Sigma = |K|$ is a triangulation, then

$$\boxed{\chi(\Sigma) := V - E + F}$$

# of vertices # of edges # of 2-dim. faces

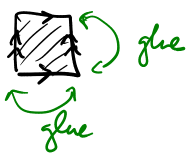
Fact if Σ and Σ' are homeomorphic, then $\chi(\Sigma) = \chi(\Sigma')$ (actually, the Euler Characteristic is an invariant of homotopy equivalence)

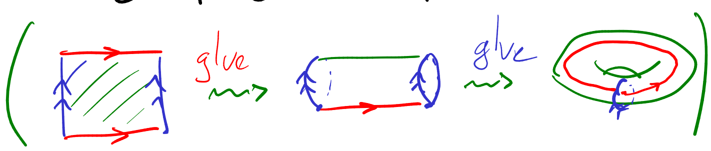
Exercise Show that $\chi(\Sigma_{g,b}) = 2 - 2g - n$

Deduce that if $\Sigma_{g,b}$ and $\Sigma_{g',b'}$ are homeomorphic then $g = g'$ and $b = b'$

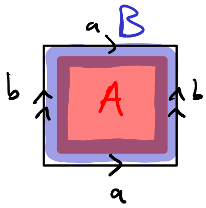
We now want to describe the fundamental group of a compact surface. Let's start with the torus $\mathbb{T}$.

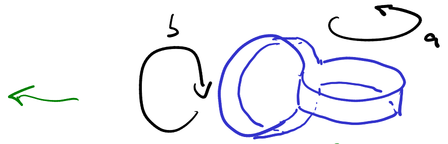
$\mathbb{T}$ can be described as





We can then use Van Kampen theorem on



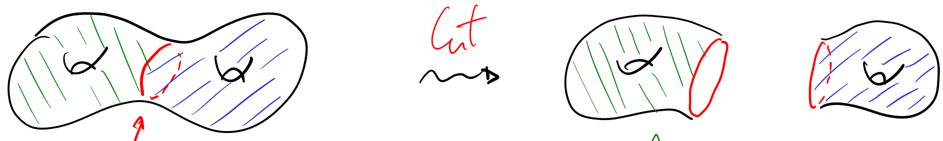


B is made of two paper strips

$\rightsquigarrow \quad \pi_1(\mathbb{T}) \cong \langle a, b \mid [a,b] \rangle$

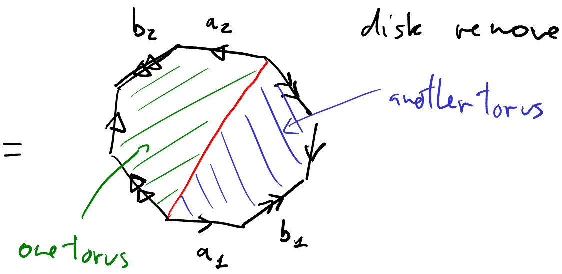
i.e. it's the quotient of the free group $F_2 = \langle a, b \rangle$ under the normal closure of the group generated by the commutator $[a,b] = aba^{-1}b^{-1} \in F_2$

Genus 2 case:



If we cut along this curve, we obtain 2 tori with one disk removed

$\rightsquigarrow \Sigma_2 =$



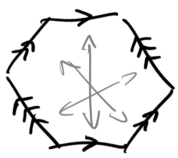
$\xrightarrow{\text{Van Kampen}}$ $\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1][a_2,b_2] \rangle$

Exercise

- Prove that $\boxed{\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1]\cdots[a_g,b_g] \rangle}$
- Prove that $\pi_1(\Sigma_{g,b}) = F_{2g+b-1}$ if $b \geq 1$

free group in $2g+b-1$ generators

Exercise Show that $\Sigma_g = (2g+2)$-gon where opposite edges are identified



§3: Surfaces have more structure (and admit Riemannian Metrics)

Let $M \cong |K|$ and $N \cong |L|$ be two triangulated manifolds. A map $F: M \to N$ is Piecewise-Linear (PL) if there is a subdivision K' of K such that the image under F of each simplex σ in K' is contained in a single simplex τ in L, and the restriction $F|_\sigma : |\sigma| \to |\tau|$ is a linear map.

Thm If $|K| \to \Sigma$ and $|L| \to \Sigma$ are two triangulations of the same surface, then they are PL-equivalent
(i.e. there exists a PL-homeomorphism $|K| \cong |L|$
(if it exists, the inverse of a PL-map is PL))

See [Moise, Theorem 4 Chap. 8] for a proof. This implies that in dimension 2 admits a unique triangulation up to PL-equivalence.
(i.e. there is a correspondence (2-dim top. mflds) $\longleftrightarrow$ (2 dim PL mflds))
This statement is called Hauptvermutung.

The Hauptvermutung holds in dimension 3 as well, but it is false in higher dimension.

Rmk It is a standard result of algebraic topology that any continuous map between simplicial complexes is homotopy-equivalent to a PL map (in fact, it is equivalent to a simplicial map) - see [Hatcher] for a proof.
Still, it need not be the case that a homeomorphism is close to a PL-homeomorphism. One of the key steps in the proof of the above theorem is to show that this fact holds for 2-dim. mflds.

The hauptvermutung allows one to prove the following:

Thm Any surface Σ admits a unique differentiable structure (up to diffeomorphism)

We don't need this to show that top. surfaces have Riem. metrics, but it's good to know

PL-mfld := n-dim manifold with triangulation s.t. the link of every vertex is an PL-sphere S^{n-1} ← this is irrelevant for $n \leq 3$

← Inductive definition

A nice summary about Top <-> PL <-> Diff is on Mathoverflow question "classification of surfaces and the Top Diff Pl categories".

In short, the story goes as follows:

the notion of compatibility is defined in terms of "smooth triangulations"

1 - Every differentiable manifold admits a unique compatible PL structure (up to PL-equival.)

2 - in general, a PL manifold need not have a compatible differentiable structure (a.k.a. "smoothing"). If it does, it need not be unique.

3 - if $\dim(M) \leq 7$, then a PL manifold does have a smoothing.

4 - if $\dim(M) \leq 6$, the smoothing is also unique (up to diffeo)

In order to find differentiable structures on a manifold, one usually starts with a combinatorial description (e.g. triangulation or handlebody decomposition) and tries to smooth it. By (1) this technique is general, because every smooth manifold can be espressed via a triangulation.

5 - there exist topological manifolds that have no triangulations, and hence no differentiable structure (e.g. E8)

6 - there exist topological manifolds that have uncountably many inequivalent PL-structures, and hence — always by (1) — uncountably many differentiable structures (e.g $\mathbb{R}^4$)

7 - in $\dim \leq 3$ the Hauptvermutung holds true: every topological manifold admits a unique PL-structure and hence a unique differentiable structure.

Some references: (1) is proved in [Munkres: El. diff. topology]
(3) & (4) for $\dim \leq 3$ is proved in [Thurston: 3-dimensional geom. and top.]

Rmk Proving (3) (and possibly (4) as well) in dim 2 (which is what we need) should be easy. It could presumably be an exercise.

It is now easy to prove the following

i.e. it has a (Riemannian) metric that induces the correct topology

Th Every surface admits a Riemannian metric.

Pf: Let Σ be a topological surface. It is triangulable by Radó's Th. Since it has dimension 2, it admits a (unique) differentiable structure.

Let $(U_i, \varphi_i)_{i\in I}$ be an atlas of differentiable local coordinates for Σ, i.e. $\varphi_i : U_i \to \mathbb{R}^2$ s.t. the change-of-coordinates are smooth. We can assume the covering $\{U_i\}_{i\in I}$ is locally finite.

The idea is that the pull-back of the Euclidean metric of $\mathbb{R}^2$ via φ_i induces a (locally defined) Riemannian metric on U_i. All we have to do then is to "smoothly merge" these Riemannian metrics as $i\in I$ varies.

Let $\rho_i : \Sigma \to \mathbb{R}_{\geq 0}$ a smooth partition of the unity subordinated to the covering $\{U_i\}_{i\in I}$

(I.e. ρ_i are smooth functions with $\overline{\text{support}(\rho_i)} \subset U_i$ and such that $\sum_{i\in I} \rho_i(p) = 1 \quad \forall p \in \Sigma$)

We define a Riemannian metric on the planar domain $\varphi_i(U_i) \subseteq \mathbb{R}^2$ as follows:

take the pull-back under the change-of coordinate map.

this is the Euclidean Riem. metric on $\varphi_j(U_j) \subseteq \mathbb{R}^2$ evaluated at the point $\psi_{ij}(p) = \varphi_j(\varphi_i^{-1}(p))$ and rescaled by $\rho_j(\varphi_i^{-1}(p))$

$$\boxed{\langle \cdot,\cdot\rangle_p := \sum_{i\in I} \psi_{ij}^{*}\left(\rho_j(\varphi_i^{-1}(p))\langle\cdot,\cdot\rangle_{Eucl}\right)}$$

Technically, this is defined only for points in $\varphi_i(U_i\cap U_j)$. We set it to 0 at every point where it is not defined. The nice thing is that this is still smooth, because we are rescaling with $\rho_j(\varphi_i^{-1}(p))$, which goes to 0 as $\varphi_i^{-1}(p)$ exits its support (i.e. p exits $\varphi_i(U_i\cap U_j)$)

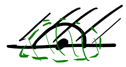
Explicitely, for $p\in \varphi_i(U_i)$ and v, w vectors based at p:

$$\boxed{\langle v,w\rangle_p := \sum_{i\in I} \rho_j(\varphi_i^{-1}(p))\langle d_p\psi_{ij}(v), d_p\psi_{ij}(w)\rangle_{Eucl}}$$

In intrinsic notation, we are defining a scalar product $\langle\cdot,\cdot\rangle_p$ on $T_p\Sigma$ letting $\langle \cdot,\cdot\rangle_p := \sum_i \rho_i(p)\ \varphi_i^{*}\langle\cdot,\cdot\rangle_{Eucl}$.
The above messy formula is a price to pay for the privilege of doing everything in local coordinates.

Since the covering $\{U_i\}_{i\in I}$ is locally finite, the above expression is (locally) a finite sum of smooth objects, and hence it is smooth.
Since the functions f_i are non-negative and for every $p \in \Sigma$ $\exists i$ $f_i(p) > 0$, it is also clear that $\langle v,v\rangle_p > 0$ $\forall p \in \varphi_i(U_i)$ and $v \in T_p\varphi_i(U_i) \setminus \{0\}$ □

Rmk

- The proof works also when $\partial\Sigma \neq \emptyset$: remember that in that case the differentiable structure sends neighborhoods of points in $\partial\Sigma$ to the half-plane 
Further, the change-of-coordinate maps are required to extend smoothly on a neighbourhood in $\mathbb{R}^2$.
Note also that we can define the Riemannian metric so that $\partial\Sigma$ is geodesic

- The same proof shows that every n-dimensional smooth manifold admits a Riemannian metric

§4 : Surfaces have complex structures

We are now going to show that every (topological) surface can be given a complex structure i.e. it can be made into a Riemann surface.

Let Σ be a topological surface with $\partial\Sigma = \emptyset$ (recall that our definition of Riemann surface has no boundary).
Then we can choose a Riemannian metric on Σ.
That is, we have charts $\varphi_i : U_i \longrightarrow \mathbb{R}^2$ and compatible Riemannian metrics on $\Omega_i := \varphi_i(U_i) \subseteq \mathbb{R}^2$.

Def a chart $\varphi : U \to \Omega \subseteq \mathbb{R}$ is said to be isothermal (φ_i is a set of isothermal local coordinates) for the Riemannian metric if the induced Riemannian metric on Ω takes the form $\boxed{\langle\cdot,\cdot\rangle_p = \lambda_p \langle\cdot,\cdot\rangle_{Eucl}}$ for some $\lambda : \Omega \to \mathbb{R}_{>0}$

Key Remark: given a chart $\varphi : U \longrightarrow \Omega \subseteq \mathbb{R}$ on a Riemannian surface Σ, the following are equiv:

- ⓐ φ is isothermal
- ⓑ the Riemannian metric that Σ induces on Ω is conformally equivalent to the Euclidean metric.

The following is a hard theorem:

(this requirement is not important)

Thm (existence of isothermal coordinates) Let (Σ, d) be a Riemann surface with $\partial\Sigma = \emptyset$.
Then, $\forall p \in \Sigma$ there exists a connected open neighbourhood $U \subset \Sigma$ and an isothermal chart $\varphi : U \to \Omega \subseteq \mathbb{R}^2$.

We are not going to prove this. The proof basically amounts to solving a PDE system (i.e. finding a solution for the Beltrami equations). You can find a proof in [Spivak - Addendum 1 to Chapter 9, Volume 4]

There is also one proof in the paper [Chern: "an elementary proof of the existence of isothermal coordinates"]

The theorem is false in $\dim > 2$.

Recall that all our surfaces are oriented and maps are orientation preserving.

It is now easy to prove the following:

Thm Let Σ be a surface with empty boundary. Then
1) Every Riemannian metric on Σ induces a unique complex structure (up to conformal equivalence)
2) If two Riemannian metrics d_1, d_2 on Σ induce the same complex structure, then they are conformally eq.

by "induce" we mean that the angles w.r.t. the complex structure coincide with the angles w.r.t the Riemannian structure

Proof. Let (Σ, d) be a Riemannian metric.

Thm $\exists$ isothermal coord $\longrightarrow$ There is an atlas $\varphi_i : U_i \xrightarrow{\cong} \Omega_i \subseteq \mathbb{R}^2$ such that the Riemannian metric on Ω_i is expressed as $\lambda_i(p)\langle \cdot, \cdot \rangle_{Eucl}$ $\quad \forall i$

We claim that (by ignoring the extra metric structure and seeing $\Omega_i \subseteq \mathbb{R}^2 = \mathbb{C}$) this atlas gives us a complex structure.

That is, we need to check that the change-of-coordinates are bi-holomorphisms.

By hypotheses $\varphi_i \circ \varphi_j^{-1} : \underbrace{\varphi_j(U_i \cap U_j)}_{\substack{\parallel \\ \text{let: } \Omega_j^i \subset \Omega_j}} \longrightarrow \underbrace{\varphi_i(U_i \cap U_j)}_{\substack{\parallel \\ \text{and: } \Omega_i^j \subseteq \Omega_i}}$ are Riemannian isometries

Thus we have:

$$(\Omega_j^i, \langle \cdot, \cdot \rangle_{Eucl}) \underset{\text{conformal equivalence}}{\overset{id}{\sim}} (\Omega_j^i, \lambda_j \langle \cdot, \cdot \rangle_{Eucl}) \xrightarrow[\substack{\text{isometry} \\ \Downarrow \\ \text{conf. eq.}}]{\varphi_i \circ \varphi_j} (\Omega_i^j, \lambda_i \langle \cdot, \cdot \rangle_{Eucl}) \underset{\text{conf. eq.}}{\overset{id}{\sim}} (\Omega_i^j, \lambda_i \langle \cdot, \cdot \rangle_{Eucl})$$

here we mean in the sense of "angle preserving"

That is, Ω_j^i, Ω_i^j are conformally equivalent w.r.t. the Euclidean metric (i.e. as domains in $\mathbb{C}$), and we know that this happens if and only if $\varphi_i \circ \varphi_j$ is a biholomorphism (Chapter 1)

The uniqueness of the complex structure is automatic.
That is, by definition we said that a complex structure is induced by a Riem. metric d if the complex angles & the Riemannian angles coincide. This is equivalent to saying that the complex charts $\varphi : U \to \Omega \subset \mathbb{C}$ are isothermal.
If we are given two sets of isothermal coordinates $\varphi_i : U_i \to \Omega_i$, $\varphi_j' : U_j' \to \Omega_j'$ for the same metric d, changing from one atlas to the other will just consist of taking some Riemannian isometries (on charts) because the metrics on Ω_i and Ω_j' are those induced from the same d on Σ.
Again, Riemannian isometries preserve the angles $\rightsquigarrow$ they are conformal eq.
$\rightsquigarrow$ bi-holomorphisms.

Part 2 is just as simple: d_1 and d_2 define the same complex structure if and only if there is a homeomorphism $\Sigma \xrightarrow{\Psi} \Sigma$ and atlases of isothermal coordinates for (Σ, d_1) and (Σ, d_2) such that Σ induces biholomorphisms between them:

$$\boxed{\begin{array}{ccccccc} (U_i,d_1) & \xrightarrow[\text{isothermal}]{\varphi_i} & \Omega_i & \xrightarrow[\text{bi-holom.}]{\varphi_j\circ\Psi\circ\varphi_i} & \Omega_j & \xleftarrow[\text{isothermal}]{\varphi_j} & (U_j,d_2)\\ & \Downarrow & & \Downarrow & & \Downarrow & \\ & \text{conf. eq.} & & \text{conf. eq.} & & \text{conf. eq.} & \end{array}}$$

That is, $\Psi : (\Sigma, d_1) \to (\Sigma, d_2)$ must preserve the angles i.e. it is conformal. $\square$

Cor every topological surface admits a complex structure.

Rmk

- It is possible to prove directly that surfaces have a complex structure without passing through differential geometry. One way to do it could be by proving directly that there exists a complex atlas by choosing a maximal set of charts so that the change-of-coordinates are bi-holomorphic (not easy)
One can develop all the theory of Riemann surfaces this way and then show that there are many meromorphic functions. The fact that Riemann surfaces are conformally equivalent to Riemannian surfaces (in the sense of angles) can be deduced from this

  Also, in this setting one need not to assume that the surface is 2^{nd} countable: it will be a consequence of the existence of the complex structure.
We did not take this approach because we are going to use more differential geometry than complex analysis

- It turns out that (closed) Riemann surfaces have some favourite Riemannian metrics (of constant curvature). We will discuss about it in the next chapter.